Effects of Ionization Statistics and Gas Gain Fluctuation on Charge Centroid

In the case of finite size pads and finite track angle

1 Charge Centroid Method

1.1 Pad Response Function and Charge on a Given Pad

We now introduce a pad row of pitch w and length L to measure the charge centroid:

$$\bar{x} \;=\; \sum_a Q_a\,(wa)/\sum_a Q_a, \tag{1}$$

where Q_a is the charge on pad a and is given as the sum of contributions from seed electrons originating from N primary ionizations followed by secondary ionizations resulting in M_i electrons for i-th primary ionization:

$$Q_a = \sum_{i=1}^{N}\sum_{j=1}^{M_{ij}} G_{ij}\,\tilde{F}_a(\tilde{x} + y_i\,\tan\phi + \Delta x_{ij},\, y_i + \Delta y_{ij}) + \Delta Q_a, \tag{2}$$

with $\tilde{F}_a(x_{ij}, y_{ij})$ being the response function of pad a for seed electron (ij) arriving at $(x_{ij}, y_{ij}) := (\tilde{x} + y_i\,\tan\phi\Delta x_{ij}, y_i + \Delta y_{ij})$ and ΔQ_a being the electronic noise on pad a. Notice that the track in question has been regarded as a straight line (good approximation locally) and parametrized as $x = \tilde{x} + y\,\tan\phi$ and hence i-th primary ionization is at $(x_i, y_i) = (\tilde{x} + y_i\,\tan\phi, y_i)$.

Notice also that the pad response function satisfies the sum rule:

$$\sum_a \tilde{F}_a(x_{ij}, y_{ij}) \;=\; R(y_{ij}), \tag{3}$$

where $R(y_{ij})$ is the fraction of charge accepted by the pad row in question. For usual readout pads, we can factorize the response function as

$$\tilde{F}_a(x_{ij}, y_{ij}) \;=\; F_a(x_{ij})\,R(y_{ij}), \tag{4}$$

and the sum rule reduces to

$$\sum_a F_a(x_{ij}) \;=\; 1. \tag{5}$$

In what follows we assume this factorization.

1.2 Probability Distribution Function for Charge on a Given Pad

The probability distribution for Q_a is then given by

$$\begin{aligned} P_a(Q_j;\tilde{x}) \;=\; & \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right.\\ & \left. \prod_{j=1}^{M_i} \left(\int_{-\infty}^{+\infty} d\Delta y_{ij} P_D(\Delta y_{ij};\sigma_d) \int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij};\sigma_d) \int d\left(\frac{G_{ij}}{\bar{G}}\right)\, P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right)\right)\right] \end{aligned}$$

$$\times \int d\Delta Q_a \, P_E(\Delta Q_a; \sigma_E) \, \delta \left(Q_a - \sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij} \, F_a(x_{ij}) R(y_{ij}) - \Delta Q_a \right), \tag{6}$$

where P_{PI}, P_{SI}, P_D, and P_G are probability distribution functions for primary ionization statistics[1], secondary ionization statistics, diffusion[2], and gain fluctuation, respectively, and P_E represents a constant electronic noise with $\langle \Delta Q_j \rangle = 0$ and $\left\langle \Delta Q_j^2 \right\rangle = \sigma_E^2$.

2 Variance of Charge Centroid

The probability distribution for the charge centroid is then given by

$$\begin{aligned} P(\bar{x}; \tilde{x}) \;=\; & \sum_{N=1}^{\infty} P_{PI}(N; \bar{N}) \prod_{i=1}^{N} \left[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right. \\ & \left. \prod_{j=1}^{M_i} \left(\int_{-\infty}^{+\infty} d\Delta y_{ij} P_D(\Delta y_{ij}; \sigma_d) \int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij}; \sigma_d) \int d\left(\frac{G_{ij}}{\bar{G}}\right) \, P_G\left(\frac{G_{ij}}{\bar{G}}; \theta\right) \right) \right] \\ & \times \int d\Delta Q_a \, P_E(\Delta Q_a; \sigma_E) \, \delta \left(Q_a - \sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij} \, F_a(x_{ij}) R(y_{ij}) - \Delta Q_a \right) \\ & \qquad \times \delta \left(\bar{x} - \frac{\sum_a Q_a \, (aw)}{\sum_a Q_a} \right). \end{aligned} \tag{7}$$

Since the probability distribution $P(\bar{x}; \tilde{x})$ depends on the true x-coordinate $\tilde{x}$ of the track at $y = 0$, we average over $\tilde{x}$ to define $\sigma_{\bar{x}}$:

$$\sigma_{\bar{x}}^2 \;\equiv\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \int d\bar{x} \, P(\bar{x}; \tilde{x}) \, (\bar{x} - \tilde{x})^2. \tag{8}$$

2.1 Separation of the Electronic Noise Contribution

Substituting Eq.(7) in this and integrating it over $\bar{x}$ and Q_a, which is trivial, we obtain

$$\begin{aligned} \sigma_{\bar{x}}^2 \;=\; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \int d\bar{x} \sum_{N=1}^{\infty} P_{PI}(N; \bar{N}) \prod_{i=1}^{N} \left[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right. \\ & \left. \prod_{j=1}^{M_i} \left(\int_{-\infty}^{+\infty} d\Delta y_{ij} P_D(\Delta y_{ij}; \sigma_d) \int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij}; \sigma_d) \int d\left(\frac{G_{ij}}{\bar{G}}\right) \, P_G\left(\frac{G_{ij}}{\bar{G}}; \theta\right) \right) \right] \\ & \times \int d\Delta Q_a \, P_E(\Delta Q_a; \sigma_E) \, \delta \left(Q_a - \sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij} \, F_a(x_{ij}) R(y_{ij}) - \Delta Q_a \right) \\ & \qquad \times \delta \left(\bar{x} - \frac{\sum_a Q_a \, (aw)}{\sum_a Q_a} \right) \, (\bar{x} - \tilde{x})^2 \\ \simeq \; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N; \bar{N}) \prod_{i=1}^{N} \left[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right. \end{aligned}$$

[1] P_{PI} is a Poisson distribution with a mean value of $\bar{N}$. Remember that $\bar{N}$ depends on the track angle $\bar{N} = \bar{n} \Delta Y \sqrt{1 + \tan^2\phi + \tan^2\lambda}$, where $\bar{n}$ is the average number of primary ionization clusters per unit track length.

[2] P_D is a Gaussian centered at zero and having a standard deviation of $\sigma_d = C_D \sqrt{z}$, where z is the drift length.

$$\prod_{j=1}^{M_i}\left(\int_{-\infty}^{+\infty} d\Delta y_{ij} P_D(\Delta y_{ij};\sigma_d)\int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij};\sigma_d)\int d\left(\frac{G_{ij}}{\bar{G}}\right)\, P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right)\right)\Bigg]$$
$$\times\int d\Delta Q_a\, P_E(\Delta Q_a;\sigma_E)\left\{\frac{\sum_a (aw)\,\left(\sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\, F_a(x_{ij})R(y_{ij})+\Delta Q_a\right)}{\sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})}-\tilde{x}\right\}^2$$

where in the last line we have used the sum rule (Eq.(5)) and ignored the electronic noise as compared with the total accepted charge by the pad row in question:

$$\begin{aligned}\sum_a Q_a &= \sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\sum_a F_a(x_{ij})\,R(y_{ij})+\sum_a \Delta Q_a\\ &= \sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\,R(y_{ij})+\sum_a\Delta Q_a\\ &\simeq \sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\,R(y_{ij}).\end{aligned}$$

With this approximation, we can separate the contribution from the electronic noise as follows:

$$\begin{aligned}\sigma_{\tilde{x}}^2 \;\simeq\; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\sum_{N=1}^{\infty} P_{PI}(N;\bar{N})\prod_{i=1}^N\Bigg[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}}\frac{dy_i}{\Delta Y}\sum_{M_i=1}^{\infty}P_{SI}(M_i)\\ &\prod_{j=1}^{M_i}\left(\int_{-\infty}^{+\infty} d\Delta y_{ij} P_D(\Delta y_{ij};\sigma_d)\int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij};\sigma_d)\int d\left(\frac{G_{ij}}{\bar{G}}\right)\, P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right)\right)\Bigg]\\ &\qquad\times\left\{\frac{\sum_a (aw)\,\sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\, F_a(x_{ij})R(y_{ij})}{\sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})}-\tilde{x}\right\}^2\\ &+\int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\sum_{N=1}^{\infty} P_{PI}(N;\bar{N})\prod_{i=1}^N\Bigg[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}}\frac{dy_i}{\Delta Y}\sum_{M_i=1}^{\infty}P_{SI}(M_i)\\ &\prod_{j=1}^{M_i}\left(\int_{-\infty}^{+\infty} d\Delta y_{ij} P_D(\Delta y_{ij};\sigma_d)\int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij};\sigma_d)\int d\left(\frac{G_{ij}}{\bar{G}}\right)\, P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right)\right)\Bigg]\\ &\qquad\times\int d\Delta Q_a\, P_E(\Delta Q_a;\sigma_E)\left\{\frac{\sum_a (aw)\Delta Q_a}{\sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})}\right\}^2,\end{aligned}$$

where the cross terms being linear in terms of ΔQ_a, which should vanish upon the ΔQ_a integration, have been dropped. By changing one of the integration variables from Δy_{ij} to $y_{ij} = y_i + \Delta y_{ij}$, we obtain

$$\begin{aligned}\sigma_{\tilde{x}}^2 \;\simeq\; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\sum_{N=1}^{\infty} P_{PI}(N;\bar{N})\prod_{i=1}^N\Bigg[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}}\frac{dy_i}{\Delta Y}\sum_{M_i=1}^{\infty}P_{SI}(M_i)\\ &\prod_{j=1}^{M_i}\left(\int_{-\infty}^{+\infty} dy_{ij} P_D(y_{ij}-y_i;\sigma_d)\int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij};\sigma_d)\int d\left(\frac{G_{ij}}{\bar{G}}\right)\, P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right)\right)\Bigg]\\ &\qquad\times\left\{\frac{\sum_a (aw)\,\sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\, F_a(\tilde{x}+y_i\tan\phi+\Delta x_{ij})R(y_{ij})}{\sum_{i=1}^N\sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})}-\tilde{x}\right\}^2\\ &+\sum_{N=1}^{\infty} P_{PI}(N;\bar{N})\prod_{i=1}^N\Bigg[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}}\frac{dy_i}{\Delta Y}\sum_{M_i=1}^{\infty}P_{SI}(M_i)\end{aligned}$$

$$\prod_{j=1}^{M_i} \left(\int_{-\infty}^{+\infty} dy_{ij} P_D(y_{ij} - y_i; \sigma_d) \int d\left(\frac{G_{ij}}{\bar{G}}\right) \; P_G\left(\frac{G_{ij}}{\bar{G}}; \theta\right)\right)\Bigg]$$
$$\times \; \sigma_E^2 \sum_a (aw)^2 \left\{ \frac{1}{\left(\sum_{i=1}^N \sum_{j=1}^{M_i} G_{ij} \, R(y_{ij})\right)^2} \right\}, \quad (9)$$

where $\tilde{x}$, Δx_{ij}, and Q_E integrations, which are trivial for the second term (the electronic noise term), have been carried out. The quantity sandwiched by the braces in the second term is the inverse square of the total charge, $Q := \sum_a Q_a$, collected by the pad row in question. The summations over N and M_i and the integrations over dy_i, dy_{ij}, and G_{ij} mean that the average of this quantity should be taken:

$$\left\langle \frac{1}{Q^2} \right\rangle := \sum_{N=1}^{\infty} P_{PI}(N; \bar{N}) \prod_{i=1}^{N} \left[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right.$$
$$\left. \prod_{j=1}^{M_i} \left(\int_{-\infty}^{+\infty} dy_{ij} P_D(y_{ij} - y_i; \sigma_d) \int d\left(\frac{G_{ij}}{\bar{G}}\right) \; P_G\left(\frac{G_{ij}}{\bar{G}}; \theta\right)\right)\right]$$
$$\times \left\{ \frac{1}{\left(\sum_{i=1}^N \sum_{j=1}^{M_i} G_{ij} \, R(y_{ij})\right)^2} \right\}. \quad (10)$$

With this definition we can now express the contribution from the electronic noise to the spatial resolution as

$$\sigma_{\bar{x},E}^2 := \sigma_E^2 \left\langle \frac{1}{Q^2} \right\rangle \sum_a (aw)^2 \quad (11)$$

2.2 The General Formula

Since the second term has been disposed of, let us now move on to further reducing the first term of Eq.(9). Be warned that we will drop the electronic noise contribution in what follows, for notation economy. The spatial resolution formula then becomes

$$\sigma_{\bar{x}}^2 \simeq \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N; \bar{N}) \prod_{i=1}^{N} \left[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right.$$
$$\left. \prod_{j=1}^{M_i} \left(\int_{-\infty}^{+\infty} dy_{ij} \, P_D(y_{ij} - y_i; \sigma_d) \int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij}; \sigma_d) \int d\left(\frac{G_{ij}}{\bar{G}}\right) \; P_G\left(\frac{G_{ij}}{\bar{G}}; \theta\right)\right)\right]$$
$$\times \left\{ \frac{\sum_a (aw) \; \sum_{i=1}^N \sum_{j=1}^{M_i} G_{ij} \, F_a(\tilde{x} + y_i \tan\phi + \Delta x_{ij}) R(y_{ij})}{\sum_{i=1}^N \sum_{j=1}^{M_i} G_{ij} \, R(y_{ij})} - \tilde{x} \right\}^2.$$

Notice here that Δx_{ij} appears only through F_a, which motivates us to define

$$\langle F_a \rangle_{\Delta x}^{x_i} := \int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij}; \sigma_d) \, F_a(x_i + \Delta x_{ij})$$
$$\langle F_a F_b \rangle_{\Delta x}^{x_i} := \int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij}; \sigma_d) \, F_a(x_i + \Delta x_{ij}) \, F_b(x_i + \Delta x_{ij}), \quad (12)$$

where the superfix x_i indicates that the left-hand sides depend only on the x-location of the i-th primary cluster after the diffusion average. Since the x-location of the i-th cluster is given by

$x_i = x_i(y_i) := \tilde{x} + y_i \tan\phi$, the y_i dependence vanishes if $\tan\phi = 0$. With these defined the above equation for the spatial resolution reduces to

$$
\begin{aligned}
\sigma_{\tilde{x}}^2 \;\simeq\; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right. \\
& \left. \prod_{j=1}^{M_i} \left(\int d\left(\frac{G_{ij}}{\bar{G}}\right) P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right) \int_{-\infty}^{+\infty} dy_{ij}\, P_D(y_{ij}-y_i;\sigma_d) \right) \right] \\
& \quad \times \left\{ \frac{\sum_{a,b}(abw^2) \sum_{i=1}^{N} \left[\langle F_a F_b \rangle_{\Delta x}^{x_i(y_i)} - \langle F_a \rangle_{\Delta x}^{x_i(y_i)} \langle F_b \rangle_{\Delta x}^{x_i(y_i)} \right] \sum_{j=1}^{M_i} (G_{ij}\, R(y_{ij}))^2}{\left(\sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij}\, R(y_{ij}) \right)^2} \right. \\
& \qquad \left. + \left(\frac{\sum_a (aw) \sum_{i=1}^{N} \langle F_a \rangle_{\Delta x}^{x_i(y_i)} \sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})}{\sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})} - \tilde{x} \right)^2 \right\} . \qquad (13)
\end{aligned}
$$

This is the most general formula, which can be numerically evaluated once the PDFs and the pad response functions are given.

3 Wide Pad Row Approximation

In order to further proceed in analytic calculations, we need to make some approximation. One such approximation is to assume that the width of the pad response function in the direction perpendicular to pad rows can be ignored compared with the pad length, L. Under this assumption, we can approximate $R(y_{ij})$ as

$$
R(y_{ij}) \;\simeq\; \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right) . \qquad (14)
$$

Notice that with this "wide pad row" approximation, we can ignore those electrons which do not make it to the pad row in question, which means that for a given set of y_{ij}'s, if we sum only over the accepted electrons in the expressions in the braces, we can set $R(y_{ij}) = 1$ and hence the y_{ij} dependence will disappear there as long as y_{ij}'s move within the pad row's acceptance. In order to take advantage of this, we insert

$$
1 \;=\; R(y_{ij}) + (1 - R(y_{ij})) \qquad (15)
$$

in front of the $P_D(y_{ij} - y_i)$ factor:

$$
\begin{aligned}
\sigma_{\tilde{x}}^2 \;\simeq\; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right. \\
& \left. \prod_{j=1}^{M_i} \left(\int d\left(\frac{G_{ij}}{\bar{G}}\right) P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right) \int_{-\infty}^{+\infty} dy_{ij}\, P_D(y_{ij}-y_i;\sigma_d)\, [R(y_{ij}) + (1 - R(y_{ij}))] \right) \right] \\
& \quad \times \left\{ \frac{\sum_{a,b}(abw^2) \sum_{i=1}^{N} \left[\langle F_a F_b \rangle_{\Delta x}^{x_i(y_i)} - \langle F_a \rangle_{\Delta x}^{x_i(y_i)} \langle F_b \rangle_{\Delta x}^{x_i(y_i)} \right] \sum_{j=1}^{M_i} (G_{ij}\, R(y_{ij}))^2}{\left(\sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij}\, R(y_{ij}) \right)^2} \right. \\
& \qquad \left. + \left(\frac{\sum_a (aw) \sum_{i=1}^{N} \langle F_a \rangle_{\Delta x}^{x_i(y_i)} \sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})}{\sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij}\, R(y_{ij})} - \tilde{x} \right)^2 \right\} . \qquad (16)
\end{aligned}
$$

Notice that, since the effect of the operation of the second term, $(1 - R(y_{ij}))$, of Eq.(15) is to remove y_{ij}-dependent terms in the expressions in the braces[3], the first (second) term of Eq.(15) corresponds to the case where electron (ij) does (does not) make it to the pad row in question. Let's assume that out of M_i secondary electrons from the i-th primary ionization, k_i electrons are accepted by the pad row in question. This corresponds to picking up k_i factors from the first term and $(M_i - k_i)$ factors from the second term out of the inserted product of unities:

$$1 \;=\; \prod_{j=1}^{M_i} \left[R(y_{ij}) + (1 - R(y_{ij})) \right].$$

Recall now that, after k_i accepted electrons are selected, there is no y_{ij} dependence left in the expressions in the braces. It then becomes straightforward to carry out the y_{ij} integration for each of such combinations. This is tantamount to making the following replacement:

$$\prod_{j=1}^{M_i} \left(\int_{-\infty}^{+\infty} dy_{ij}\, P_D(y_{ij} - y_i; \sigma_d) \left[R(y_{ij}) + (1 - R(y_{ij})) \right] \right) \;\to\; \sum_{k_i=0}^{M_i} {}_{M_i}C_{k_i}\, \eta(y_i)^{k_i} (1 - \eta(y_i))^{M_i - k_i},$$

where we have defined

$$\eta(y_i) \;:=\; \int_{-\infty}^{+\infty} dy_{ij}\, P_D(y_{ij} - y_i; \sigma_d)\, R(y_{ij}), \tag{17}$$

which is none other than the probability of an electron originating from the i-th primary ionization cluster created at $y = y_i$ reaching the pad row in question[4]. With this replacement Eq.(16) becomes

$$\begin{aligned}
\sigma_{\bar{x}}^2 \;&\simeq\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N; \bar{N}) \prod_{i=1}^{N} \Bigg[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \\
&\qquad \sum_{k_i=0}^{M_i} {}_{M_i}C_{k_i}\, \eta(y_i)^{k_i} (1 - \eta(y_i))^{M_i - k_i} \prod_{j=1}^{k_i} \left(\int d\left(\frac{G_{ij}}{\bar{G}}\right) P_G\left(\frac{G_{ij}}{\bar{G}}; \theta\right) \right) \Bigg] \\
&\qquad \times \left\{ \frac{\sum_{a,b} (abw^2) \sum_{i=1}^{N} \left[\langle F_a F_b \rangle_{\Delta x}^{x_i(y_i)} - \langle F_a \rangle_{\Delta x}^{x_i(y_i)} \langle F_b \rangle_{\Delta x}^{x_i(y_i)} \right] \sum_{j=1}^{k_i} (G_{ij})^2}{\left(\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij} \right)^2} \right. \\
&\qquad\qquad \left. + \left(\frac{\sum_a (aw) \sum_{i=1}^{N} \langle F_a \rangle_{\Delta x}^{x_i(y_i)} \sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} - \tilde{x} \right)^2 \right\}. \\
&\simeq\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N; \bar{N}) \prod_{i=1}^{N} \Bigg[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{k_i=0}^{\infty} \\
&\qquad \sum_{M_i=1}^{\infty} P_{SI}(M_i)\, {}_{M_i}C_{k_i}\, \eta(y_i)^{k_i} (1 - \eta(y_i))^{M_i - k_i} \prod_{j=1}^{k_i} \left(\int d\left(\frac{G_{ij}}{\bar{G}}\right) P_G\left(\frac{G_{ij}}{\bar{G}}; \theta\right) \right) \Bigg] \\
&\qquad \times \left\{ \frac{\sum_{a,b} (abw^2) \sum_{i=1}^{N} \left[\langle F_a F_b \rangle_{\Delta x}^{x_i(y_i)} - \langle F_a \rangle_{\Delta x}^{x_i(y_i)} \langle F_b \rangle_{\Delta x}^{x_i(y_i)} \right] \sum_{j=1}^{k_i} (G_{ij})^2}{\left(\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij} \right)^2} \right.
\end{aligned}$$

[3] This is a consequence of a trivial identity: $R(y_{ij})(1 - R(y_{ij})) = 0$ derived from Eq.(14).

[4] Notice that $\eta(y_i) \to R(y_i)$ in the $\sigma_d \to 0$ limit, where the PDF for the diffusion in the y direction becomes a delta function: $P_D(y_{ij} - y_i; \sigma_d) \to \delta(y_{ij} - y_i)$.

$$+\left(\frac{\sum_a (aw)\ \sum_{i=1}^{N} \langle F_a\rangle_{\Delta x}^{x_i(y_i)}\ \sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} - \tilde{x}\right)^2\Bigg\}. \tag{18}$$

Denoting

$$\bar{P}_{SI}(k_i, y_i) \ := \ \sum_{M_i=1}^{\infty} P_{SI}(M_i)\ {}_{M_i}C_{k_i}\ \eta(y_i)^{k_i}(1-\eta(y_i))^{M_i-k_i} \tag{19}$$

and exchanging the order of M_i summation and k_i summation[5], we obtain

$$\begin{aligned}\sigma_{\bar{x}}^2 \ \simeq\ & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N}\left[\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \sum_{k_i=0}^{\infty} \bar{P}_{SI}(k_i,y_i) \prod_{j=1}^{k_i}\left(\int d\left(\frac{G_{ij}}{\bar{G}}\right) P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right)\right)\right] \\ & \times\left\{\frac{\sum_{a,b}(abw^2)\ \sum_{i=1}^{N}\left[\langle F_aF_b\rangle_{\Delta x}^{x_i(y_i)} - \langle F_a\rangle_{\Delta x}^{x_i(y_i)}\langle F_b\rangle_{\Delta x}^{x_i(y_i)}\right]\sum_{j=1}^{k_i}(G_{ij})^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}\right)^2}\right. \\ & \left. + \left(\frac{\sum_a (aw)\ \sum_{i=1}^{N} \langle F_a\rangle_{\Delta x}^{x_i(y_i)}\ \sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} - \tilde{x}\right)^2\right\}.\end{aligned}$$

Let us now exchange the order of y_i integration and k_i summation and carry out the y_i integration noting that y_i appears only through $\langle F_aF_b\rangle_{\Delta x}^{x_i(y_i)}$ and $\langle F_a\rangle_{\Delta x}^{x_i(y_i)}$:

$$\begin{aligned}\sigma_{\bar{x}}^2 \ \simeq\ & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N}\left[\sum_{k_i=0}^{\infty}\int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \bar{P}_{SI}(k_i,y_i) \prod_{j=1}^{k_i}\left(\int d\left(\frac{G_{ij}}{\bar{G}}\right) P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right)\right)\right] \\ & \times\left\{\frac{\sum_{a,b}(abw^2)\ \sum_{i=1}^{N}\left[\langle F_aF_b\rangle_{\Delta x}^{x_i(y_i)} - \langle F_a\rangle_{\Delta x}^{x_i(y_i)}\langle F_b\rangle_{\Delta x}^{x_i(y_i)}\right]\sum_{j=1}^{k_i}(G_{ij})^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}\right)^2}\right. \\ & + \frac{\sum_{a,b}(abw^2)\ \sum_{i=1}^{N}\langle F_a\rangle_{\Delta x}^{x_i(y_i)}\langle F_b\rangle_{\Delta x}^{x_i(y_i)}\left(\sum_{j=1}^{k_i} G_{ij}\right)^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}\right)^2} \\ & + \frac{\sum_{a,b}(abw^2)\ \sum_{i\neq i'}\langle F_a\rangle_{\Delta x}^{x_i(y_i)}\langle F_b\rangle_{\Delta x}^{x_i(y_{i'})}\left(\sum_{j=1}^{k_i} G_{ij}\right)\left(\sum_{j'=1}^{k_{i'}} G_{i'j'}\right)}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}\right)^2} \\ & \left. -2\tilde{x}\frac{\sum_a (aw)\ \sum_{i=1}^{N} \langle F_a\rangle_{\Delta x}^{x_i(y_i)}\ \sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} + \tilde{x}^2\right\}.\end{aligned} \tag{20}$$

Defining the following new quantities:

$$\begin{aligned}\bar{\bar{P}}_{SI}(k_i) \ &:=\ \int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y}\bar{P}_{SI}(k_i,y_i) \\ \bar{\bar{P}}_{SI}(k_i,y_i) \ &:=\ \frac{1}{\bar{\bar{P}}_{SI}(k_i)}\bar{P}_{SI}(k_i,y_i) \\ \langle\langle F_aF_b\rangle\rangle^{k_i} \ &:=\ \int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y}\bar{\bar{P}}_{SI}(k_i,y_i)\ \langle F_aF_b\rangle_{\Delta x}^{x_i(y_i)}\end{aligned}$$

[5] Notice that Eq.(14) implies that $\bar{P}_{SI}(k_i,y_i) \to P_{SI}(k_i)\theta\left(\frac{L}{2}+y_{ij}\right)\ \theta\left(\frac{L}{2}-y_{ij}\right)$ in the $\sigma_d \to 0$ limit.

$$
\begin{aligned}
\langle\langle F_a\rangle\,\langle F_b\rangle\rangle^{k_i} &:= \int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y}\,\bar{\bar{P}}_{SI}(k_i,y_i)\,\langle F_a\rangle_{\Delta x}^{x_i(y_i)}\,\langle F_b\rangle_{\Delta x}^{x_i(y_i)} \\
\langle\langle F_a\rangle\rangle^{k_i} &:= \int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y}\,\bar{\bar{P}}_{SI}(k_i,y_i)\,\langle F_a\rangle_{\Delta x}^{x_i(y_i)}
\end{aligned}
\tag{21}
$$

and substituting them[6] in Eq.(20), Eq.(20) becomes

$$
\begin{aligned}
\sigma_{\bar{x}}^2 \simeq & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N}\left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i) \prod_{j=1}^{k_i}\left(\int d\left(\frac{G_{ij}}{\bar{G}}\right) P_G\left(\frac{G_{ij}}{\bar{G}};\theta\right)\right)\right] \\
& \times\left\{\frac{\sum_{a,b}(abw^2)\,\sum_{i=1}^{N}\left[\langle\langle F_aF_b\rangle\rangle^{k_i}-\langle\langle F_a\rangle\,\langle F_b\rangle\rangle^{k_i}\right]\,\sum_{j=1}^{k_i}(G_{ij})^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i}G_{ij}\right)^2}\right. \\
& +\frac{\sum_{a,b}(abw^2)\,\sum_{i=1}^{N}\left[\langle\langle F_a\rangle\,\langle F_b\rangle\rangle^{k_i}-\langle\langle F_a\rangle\rangle^{k_i}\,\langle\langle F_b\rangle\rangle^{k_i}\right]\,\left(\sum_{j=1}^{k_i}G_{ij}\right)^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i}G_{ij}\right)^2} \\
& +\frac{\sum_{a,b}(abw^2)\,\sum_{i=1}^{N}\langle\langle F_a\rangle\rangle^{k_i}\,\langle\langle F_b\rangle\rangle^{k_i}\,\left(\sum_{j=1}^{k_i}G_{ij}\right)^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i}G_{ij}\right)^2} \\
& +\frac{\sum_{a,b}(abw^2)\,\sum_{i\neq i'}\langle\langle F_a\rangle\rangle^{k_i}\,\langle\langle F_b\rangle\rangle^{k_i'}\,\left(\sum_{j=1}^{k_i}G_{ij}\right)\left(\sum_{j'=1}^{k_{i'}}G_{i'j'}\right)}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i}G_{ij}\right)^2} \\
& \left.-2\tilde{x}\,\frac{\sum_a(aw)\,\sum_{i=1}^{N}\langle\langle F_a\rangle\rangle^{k_i}\,\sum_{j=1}^{k_i}G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i}G_{ij}}+\tilde{x}^2\right\}.
\end{aligned}
$$

$$
\begin{aligned}
\simeq & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N}\left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i)\right] \\
& \times\left\{\sum_{a,b}(abw^2)\sum_{i=1}^{N}\left[\langle\langle F_aF_b\rangle\rangle^{k_i}-\langle\langle F_a\rangle\,\langle F_b\rangle\rangle^{k_i}\right]\left\langle\frac{\sum_{j=1}^{k_i}(G_{ij})^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i}G_{ij}\right)^2}\right\rangle_G^{k_i,\,\sum_{i=1}^{N}k_i}\right. \\
& +\sum_{a,b}(abw^2)\sum_{i=1}^{N}\left[\langle\langle F_a\rangle\,\langle F_b\rangle\rangle^{k_i}-\langle\langle F_a\rangle\rangle^{k_i}\,\langle\langle F_b\rangle\rangle^{k_i}\right]\left\langle\frac{\left(\sum_{j=1}^{k_i}G_{ij}\right)^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i}G_{ij}\right)^2}\right\rangle_G^{k_i,\,\sum_{i=1}^{N}k_i} \\
& +\sum_{a,b}(abw^2)\sum_{i=1}^{N}\langle\langle F_a\rangle\rangle^{k_i}\,\langle\langle F_b\rangle\rangle^{k_i}\left\langle\frac{\left(\sum_{j=1}^{k_i}G_{ij}\right)^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i}G_{ij}\right)^2}\right\rangle_G^{k_i,\,\sum_{i=1}^{N}k_i} \\
& +\sum_{a,b}(abw^2)\sum_{i\neq i'}\langle\langle F_a\rangle\rangle^{k_i}\,\langle\langle F_b\rangle\rangle^{k_i'}\left\langle\frac{\left(\sum_{j=1}^{k_i}G_{ij}\right)\left(\sum_{j'=1}^{k_{i'}}G_{i'j'}\right)}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i}G_{ij}\right)^2}\right\rangle_G^{k_i,k_{i'}\,\sum_{i=1}^{N}k_i}
\end{aligned}
$$

[6]It is instructive to note that the binomial factor in the definition (see Eq.(19)) of $\bar{P}_{SI}(k_i,y_i)$ can be approximated by a Gaussian with a mean value of $M_i\eta(y_i)$ and a r.m.s. value of $\sqrt{M_i\eta(y_i)}$ in the large M_i limit. If M_i is sufficiently large, it can further be approximated by a delta function, $\delta(k_i - M_i\eta(y_i)$, reducing $\bar{P}_{SI}(k_i,y_i)$ to $P_{SI}(k_i/\eta(y_i))$. Since the PDF of secondary ionization, $P_{SI}(M_i)$ obeys a power-law behavior, $P_{SI}(M_i)\propto M_i^{-\alpha}$ with $\alpha\simeq 2>0$, in the large M_i limit, the shape of $\bar{P}_{SI}(k_i,y_i)$ and consequently $\bar{\bar{P}}_{SI}(k_i,y_i)$ also become independent of k_i and behave like $\bar{\bar{P}}_{SI}(k_i,y_i)\propto(\eta(y_i))^{\alpha}$.

$$-2\tilde{x}\ \sum_a (aw) \sum_{i=1}^N \langle\langle F_a \rangle\rangle^{k_i} \left\langle \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right\rangle_G^{k_i,\ \sum_{i=1}^N k_i} + \tilde{x}^2 \Bigg\} .$$

The last line reads

$$\begin{aligned}
\sigma_{\bar{x}}^2 \ \simeq\ & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i)\right] \\
& \times \Bigg\{ \sum_{a,b} (abw^2) \sum_{i=1}^N \left[\langle\langle F_a F_b \rangle\rangle^{k_i} - \langle\langle F_a \rangle \langle F_b \rangle\rangle^{k_i} \right] \left\langle \frac{\sum_{j=1}^{k_i} (G_{ij})^2}{\left(\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}\right)^2} \right\rangle_G^{k_i,\ \sum_{i=1}^N k_i} \\
& + \sum_{a,b} (abw^2) \sum_{i=1}^N \left[\langle\langle F_a \rangle \langle F_b \rangle\rangle^{k_i} - \langle\langle F_a \rangle\rangle^{k_i} \langle\langle F_b \rangle\rangle^{k_i} \right] \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{k_i,\ \sum_{i=1}^N k_i} \\
& + \sum_{a,b} (abw^2) \sum_{i=1}^N \sum_{i'=1}^N \langle\langle F_a \rangle\rangle^{k_i} \langle\langle F_b \rangle\rangle^{k'_i} \left[\left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right) \left(\frac{\sum_{j'=1}^{k_{i'}} G_{i'j'}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right) \right\rangle_G^{k_i, k_{i'}\ \sum_{i=1}^N k_i} \right. \\
& \left. - \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right) \right\rangle_G^{k_i,\ \sum_{i=1}^N k_i} \left\langle \left(\frac{\sum_{j'=1}^{k_{i'}} G_{i'j'}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right) \right\rangle_G^{k'_i,\ \sum_{i=1}^N k_i} \right] \\
& + \left(\sum_a (aw) \sum_{i=1}^N \langle\langle F_a \rangle\rangle^{k_i} \left\langle \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right\rangle_G^{k_i,\ \sum_{i=1}^N k_i} - \tilde{x} \right)^2 \Bigg\} . \qquad (22)
\end{aligned}$$

The first term in the braces corresponds to the fluctuations of signal charges on two pads, which are correlated geometrically due to the finite width of the pad response function through the fluctuations of the arrival positions of individual electrons thereby being proportional to the mean square relative avalanche charges of the individual electrons. We will show later that upon the $\tilde{x}$-average this term reduces exactly to the $[B]$ term of the formula for normal incidence and hence becomes independent of $\tan\phi$. Notice that this term remains even if there is no gas gain fluctuation. The second term in the braces, on the other hand, is proportional to the mean square relative avalanche charges of the individual primary ionization clusters and is coming from the geometrical correlation due to the electrons originating from the same primary clusters. This term, hereafter called the $[D]$ term, goes away in the $\tan\phi = 0$ limit but stays finite even if there is no gas gain fluctuation as long as $\tan\phi \neq 0$. Notice that the structure of this term is very similar to the first term as is easily seen by replacing the role of $\langle F_a \rangle$ with that of F_a (see Eq.(39) of Appendix A). The third term in the braces disappears if there is no gas gain fluctuation and hence stems from the correlation through the fluctuations of the mean square relative avalanche charges of the individual primary ionization clusters. This term, hereafter called the $[G]$ term, also vanishes in the $\tan\phi = 0$ limit. The last term in the braces coincides with the $[A]$ term of the formula for normal incidence in the $\tan\phi = 0$ limit and hence it is of almost purely geometric origin. In the declustering limit (where only $k_i = 1$ contributes) it becomes indeed purely geometric even if $\tan\phi \neq 0$. It also becomes purely geometric in the zero diffusion ($\sigma_d \to 0$) limit[7]. Being constrained from the both sides, this "would-be [A] term" is thus almost purely geometric.

[7] In the zero diffusion limit under the assumption of Eq.(14), we have

$$\bar{\bar{P}}_{SI}(k_i, y_i) \to \left(\frac{\Delta Y}{L}\right) \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)$$

For practical numerical calculations, it is useful to note that

$$\left\langle \frac{\sum_{j=1}^{k_i}(G_{ij})^2}{\left(\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}\right)^2} \right\rangle_G^{k_i,\,\sum_{i=1}^{N}k_i} = k_i \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N}k_i}$$

$$\left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{k_i,\,\sum_{i=1}^{N}k_i} = k_i \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N}k_i} + (k_i^2 - k_i) \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right) \left(\frac{G_{ij'}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right) \right\rangle_{G,\,j\neq j'}^{\sum_{i=1}^{N}k_i}$$

$$\left\langle \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right\rangle_G^{k_i,\,\sum_{i=1}^{N}k_i} = k_i \left\langle \frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right\rangle_G^{\sum_{i=1}^{N}k_i}$$

and

$$\left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right) \left(\frac{\sum_{j'=1}^{k_{i'}} G_{i'j'}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right) \right\rangle_{G,i\neq i'}^{k_i,\,k_{i'}\ \sum_{i=1}^{N}k_i} = k_i\,k_{i'} \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right) \left(\frac{G_{ij'}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right) \right\rangle_{G,j\neq j'}^{\sum_{i=1}^{N}k_i},$$

where the gain-averaged quantities on the right hand sides depend only on the total number accepted electrons.

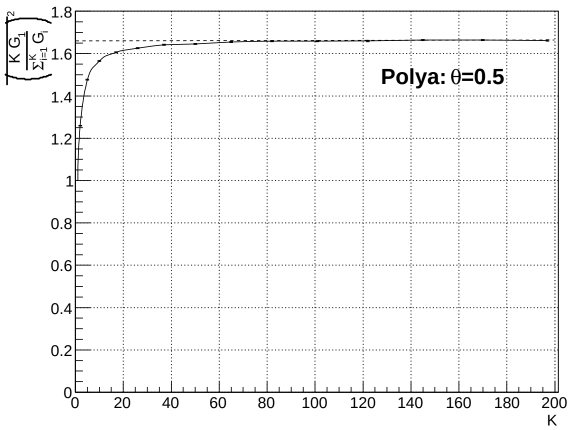

Figure 1: Relative gain variation for a single electron as a function of the number of accepted electrons ($K = \sum k_i$) by the pad row in question, assuming that gain fluctuation is Polya-distributed with $\theta = 0.5$. The average variation quickly approaches its asymptotic value of $(2+\theta)/(1+\theta) = 1.67$.

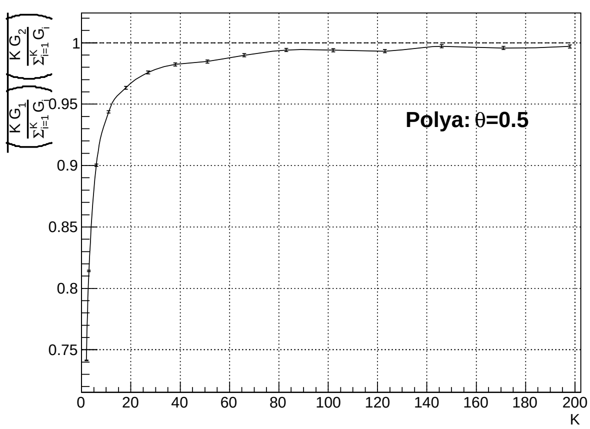

Figure 2: Correlation of relative gain fluctuations for two electrons as a function of the number of accepted electrons ($K = \sum k_i$) by the pad row in question, assuming that gain fluctuation is Polya-distributed with $\theta = 0.5$. The correlation quickly disappears as K increases.

as pointed out in foot note 5, which is indeed independent of k_i.

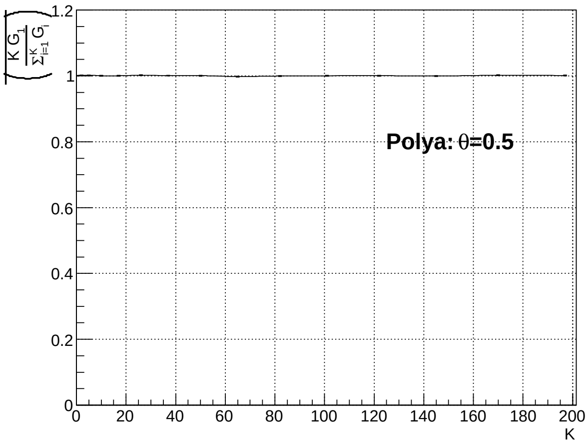

Figure 3: Average relative gain for a single electron as a function of the number of accepted electrons ($K = \sum k_i$) by the pad row in question, assuming that gain fluctuation is Polya-distributed with $\theta = 0.5$. The average is very close to 1 everywhere.

It can be shown that the gas gain fluctuation is practically negligible in the last term (the would-be [A] term) in the braces of Eq.(22), since the double average, $\langle\langle F_a \rangle\rangle^{k_i}$, is virtually independent of k_i in most practice and the numerator and the denominator in the G average cancel upon summation over the primary clusters. It can be also shown that the $[G]$ term is negligible in practice by the same token.

4 Interpretation of the Formula

The last form of the resolution formula shown in Eq.(22) comes in handy for software implementation and hence numerical calculations. For the extraction of physical meanings of each term of the formula, however, it is not necessarily the best form. For this purpose the following form is probably better:

$$
\begin{aligned}
\sigma^2_{\bar{x}} \simeq & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i) \right] \\
& \times \left\{ \sum_{i=1}^{N} \sum_{j=1}^{k_i} \left\langle \left\langle \left(\sum_a (aw)\, F_a - \sum_a (aw)\, \langle F_a \rangle_{\Delta x}^{x_i(y)} \right)^2 \right\rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N} k_i} \right. \\
& \quad + \sum_{i=1}^{N} \left\langle \left(\sum_a (aw)\, \langle F_a \rangle_{\Delta x}^{x_i(y)} - \sum_a (aw) \left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \right)^2 \right\rangle_y \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{k_i,\, \sum_{i=1}^{N} k_i} \\
& \quad + \left\langle \left(\sum_{i=1}^{N} \sum_a (aw) \left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right. \right. \\
& \qquad \qquad \left. \left. \left. - \sum_{i=1}^{N} \sum_a (aw) \left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \left\langle \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right\rangle_G^{k_i,\, \sum_{i=1}^{N} k_i} \right)^2 \right\rangle_G \right.
\end{aligned}
$$

$$+\left(\sum_a (aw)\sum_{i=1}^{N}\left\langle \langle F_a\rangle_{\Delta x}^{x_i(y)}\right\rangle_y^{k_i}\left\langle \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}}\right\rangle_G^{k_i,\,\sum_{i=1}^{N} k_i} - \tilde{x}\right)^2\Bigg\}. \tag{23}$$

Using this form, let us reinterpret the meanings of individual terms.

4.1 The [B] Term

We start from the first line in the braces, which corresponds to the $[B]$ term:

$$\begin{aligned}[B] \;&:=\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\sum_{N=1}^{\infty} P_{PI}(N;\bar{N})\prod_{i=1}^{N}\left[\sum_{k_i=0}^{\infty}\bar{\bar{P}}_{SI}(k_i)\right] \\ &\times\left\{\sum_{i=1}^{N}\sum_{j=1}^{k_i}\left\langle\left\langle\left(\sum_a (aw)\,F_a - \sum_a (aw)\,\langle F_a\rangle_{\Delta x}^{x_i(y)}\right)^2\right\rangle_{\Delta x}^{x_i(y)}\right\rangle_y^{k_i}\left\langle\left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}}\right)^2\right\rangle_G^{\sum_{i=1}^{N} k_i}\right\}.\end{aligned}$$

Notice that

$$\sum_a (aw)\,F_a = \sum_a (aw)\,F_a(x_{ij}) = \sum_a (aw)\,F_a(\tilde{x} + y_i\,\tan\phi + \Delta x_{ij})$$

gives the coordinate estimate as obtained from a single electron, which is labeled by (i,j). In the narrow pad limit ($w/\sigma_{PR} \to 0$) this becomes

$$\begin{aligned}\lim_{w/\sigma_{PR}\to 0}\sum_a (aw)\,F_a(x_{ij}) &= \lim_{w/\sigma_{PR}\to 0}\sum_a (aw)\int_{\left(a-\frac{1}{2}\right)w}^{\left(a+\frac{1}{2}\right)w}\frac{dx'}{\sqrt{2\pi}\sigma_{PR}}\exp\left[-\frac{1}{2}\left(\frac{x'-x_{ij}}{\sigma_{PR}}\right)^2\right] \\ &= \lim_{w/\sigma_{PR}\to 0}\sum_a \frac{w}{\sqrt{2\pi}\sigma_{PR}}\exp\left[-\frac{1}{2}\left(\frac{aw-x_{ij}}{\sigma_{PR}}\right)^2\right](aw) \\ &= \int_{-\infty}^{+\infty}\frac{dx'}{\sqrt{2\pi}\sigma_{PR}}\exp\left[-\frac{1}{2}\left(\frac{x'-x_{ij}}{\sigma_{PR}}\right)^2\right]x' \\ &= x_{ij} \;=\; \tilde{x} + y_i\,\tan\phi + \Delta x_{ij},\end{aligned}$$

which also implies

$$\begin{aligned}\lim_{w/\sigma_{PR}\to 0}\sum_a (aw)\,\langle F_a(x_{ij})\rangle_{\Delta x_{ij}}^{x_i(y_i)} \;&=\; \langle x_{ij}\rangle_{\Delta x_{ij}}^{x_i(y_i)} \;=\; \tilde{x} + y_i\,\tan\phi \\ \lim_{w/\sigma_{PR}\to 0}\sum_a (aw)\,\left\langle\langle F_a(x_{ij})\rangle_{\Delta x_{ij}}^{x_i(y_i)}\right\rangle_{y_i}^{k_i} \;&=\; \left\langle\langle x_{ij}\rangle_{\Delta x_{ij}}^{x_i(y_i)}\right\rangle_{y_i}^{k_i} \;=\; \tilde{x},\end{aligned}$$

since $\langle\Delta x_{ij}\rangle_{\Delta x_{ij}} = 0$ and $\langle y_i\rangle_{y_i} = 0$. In the narrow pad limit, the double average in the first line in the braces of Eq.(23) then reduces to σ_d^2, becoming independent of k_i, and hence we find

$$\text{the } [B] \text{ term } \;\to\; \sigma_d^2\left\langle\sum_{i=1}^{N}\sum_{j=1}^{k_i}\left\langle\left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}}\right)^2\right\rangle_G^{\sum_{i=1}^{N} k_i}\right\rangle_{N,k}. \tag{24}$$

Since the gain average is independent of values of individual k_i's but only dependent on their sum, this reduces to

$$\text{the } [B] \text{ term } \;\to\; \sigma_d^2\left\langle\sum_{i=1}^{N} k_i\left\langle\left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}}\right)^2\right\rangle_G^{\sum_{i=1}^{N} k_i}\right\rangle_{N,k}.$$

Using the approximation:

$$\bar{G} \simeq \frac{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} k_i}, \tag{25}$$

we can cast this in the form:

$$\text{the } [B] \text{ term} \rightarrow \sigma_d^2 \left\langle \frac{1}{\sum_{i=1}^{N} k_i} \right\rangle_{N,k} \left\langle \left(\frac{G}{\bar{\bar{G}}}\right)^2 \right\rangle_G .$$

Recalling the definition of N_{eff}:

$$N_{eff} := \left[\left\langle \frac{1}{\sum_{i=1}^{N} k_i} \right\rangle_{N,k} \left\langle \left(\frac{G}{\bar{\bar{G}}}\right)^2 \right\rangle_G \right]^{-1} . \tag{26}$$

this is exactly what we expect for the $[B]$ term in the narrow pad limit. The derivation implies that more accurate formula for the N_{eff} is

$$N_{eff} := \left[\left\langle \sum_{i=1}^{N} k_i \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N} k_i} \right\rangle_{N,k} \right]^{-1} . \tag{27}$$

It is probably useful to investigate the short and the long drift limits here. In the short drift limit, the diffusion contribution vanishes ($\sigma_d \rightarrow 0$), implying $\Delta x \rightarrow 0$, and hence the $[B]$ term vanishes as well.

In order to determine the long distance behavior of the $[B]$ term let us further process it as follows:

$$\begin{aligned}
[B] &= \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i) \right] \\
&\quad \times \left\{ \sum_{i=1}^{N} \sum_{j=1}^{k_i} \left\langle \left\langle \left(\sum_a (aw)\, F_a - \sum_a (aw)\, \langle F_a \rangle_{\Delta x}^{x_i(y)} \right)^2 \right\rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N} k_i} \right\} \\
&= \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i) \right] \times \left\{ \sum_{i=1}^{N} \sum_{j=1}^{k_i} \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N} k_i} \right. \\
&\quad \left. \times \left\langle \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(x_i+\Delta x) - \sum_a (aw)\, \langle F_a(x_i+\Delta x) \rangle_{\Delta x}^{x_i} \right)^2 \right\rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \right\}
\end{aligned}$$

Now as proved in Appendix B (B.1) one can show that the result of the $(\tilde{x}/w)$-integral in the last line is actually independent of y_i and hence we can replace x_i with $\tilde{x}$:

$$\begin{aligned}
\tilde{x}/w \text{ integral} &= \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(x_i+\Delta x) - \sum_a (aw)\, \langle F_a(x_i+\Delta x) \rangle_{\Delta x}^{x_i} \right)^2 \right\rangle_{\Delta x}^{x_i = \tilde{x} + y\tan\phi} \\
&= \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\, \langle F_a(\tilde{x}+\Delta x) \rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}} .
\end{aligned}$$

This has exactly the same form as the $[B]$ term for a perpendicular track and is indeed independent of y, which makes the y-average trivial and independent of k_i:

$$
\begin{aligned}
[B] &= \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i) \right] \times \left[\sum_{i=1}^{N} \sum_{j=1}^{k_i} \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N} k_i} \right] \\
&\quad \times \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x) \rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}} \\
&= \left[\left\langle \sum_{i=1}^{N} k_i \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N} k_i} \right\rangle_{N,k} \right] \\
&\quad \times \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x) \rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}} \\
&= \frac{1}{N_{eff}} \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x) \rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}}
\end{aligned}
$$

It is remarkable that the resultant formula of the $[B]$ term for inclined tracks coincides the one for perpendicular tracks:

$$
[B] = \frac{1}{N_{eff}} \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x) \rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}}
$$

with

$$
N_{eff} := \left[\left\langle \sum_{i=1}^{N} k_i \left\langle \left(\frac{G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N} k_i} \right\rangle_{N,k} \right]^{-1}. \tag{28}
$$

For the analysis of the long distance behavior, we can hence follow exactly the same step we used for perpendicular tracks:

$$
\begin{aligned}
\lim_{w/\sigma_d \to 0} [B] &= \lim_{w/\sigma_d \to 0} \frac{1}{N_{eff}} \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x) \rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}} \\
&= \lim_{w/\sigma_d \to 0} \frac{1}{N_{eff}} \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \\
&\quad \times \left\langle \left[\left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - (\tilde{x}+\Delta x) \right) - \left(\sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x) \rangle_{\Delta x}^{\tilde{x}} - \tilde{x} \right) + \Delta x \right]^2 \right\rangle_{\Delta x}^{\tilde{x}} \\
&= \lim_{w/\sigma_d \to 0} \frac{1}{N_{eff}} \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left[\left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - (\tilde{x}+\Delta x) \right) + \Delta x \right]^2 \right\rangle_{\Delta x}^{\tilde{x}} \\
&= \lim_{w/\sigma_d \to 0} \frac{1}{N_{eff}} \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left[\left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - (\tilde{x}+\Delta x) \right)^2 \right. \right.
\end{aligned}
$$

$$+2\,(\Delta x)\,\left(\sum_a (aw)\,F_a(\tilde{x}+\Delta x) - (\tilde{x}+\Delta x)\right) + (\Delta x)^2\Bigg]\Bigg\rangle_{\Delta x}^{\tilde{x}}$$

$$= \lim_{w/\sigma_d\to 0}\;\frac{1}{N_{eff}}\;\Bigg[\Bigg\langle \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\,F_a(\tilde{x}+\Delta x) - (\tilde{x}+\Delta x)\right)^2$$

$$+2\,(\Delta x)\,\int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\,F_a(\tilde{x}+\Delta x) - (\tilde{x}+\Delta x)\right)\Bigg\rangle_{\Delta x}^{\tilde{x}} + \sigma_d^2\;\Bigg]$$

$$= \lim_{w/\sigma_d\to 0}\;\frac{1}{N_{eff}}\;\Bigg[\Bigg\langle \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\,F_a(\tilde{x}+\Delta x) - (\tilde{x}+\Delta x)\right)^2\Bigg\rangle_{\Delta x}^{\tilde{x}} + \sigma_d^2\;\Bigg]$$

$$= \frac{1}{N_{eff}}\;\left[\int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\,F_a(\tilde{x}) - \tilde{x}\right)^2 + \sigma_d^2\;\right]$$

where in the third step use has been made of

$$\lim_{w/\sigma_d\to 0}\sum_a (aw)\;\langle F_a(\tilde{x}+\Delta x)\rangle_{\Delta x} \;=\; \lim_{w/\sigma_d\to 0}\sum_a (aw)\,F_a(\tilde{x};\sqrt{\sigma_{PR}^2+\sigma_d^2}) \;=\; \tilde{x}$$

and in the last two steps:

$$\int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\,F_a(\tilde{x}+\Delta x) - (\tilde{x}+\Delta x)\right) \;=\; 0$$

and

$$\int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\,F_a(\tilde{x}+\Delta x) - (\tilde{x}+\Delta x)\right)^2 \;=\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\,F_a(\tilde{x}) - \tilde{x}\right)^2,$$

whose proofs are given in Appendix B (B.2 and B.3).

Recalling that the $[A]$ term at $z=0$ for perpendicular tracks is given by

$$[A](z=0) \;=\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\;\langle F_a(\tilde{x}+\Delta x)\rangle_{\Delta x}^{\tilde{x}} - \tilde{x}\right)^2\Bigg|_{z=0}$$

$$= \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\,F_a(\tilde{x}) - \tilde{x}\right)^2,$$

we finally arrive at the following asymptotic form of the $[B]$ term in the long drift region:

$$\lim_{w/\sigma_d\to 0}[B] \;=\; \frac{1}{N_{eff}}\;\left[[A](z=0) + \sigma_d^2\;\right]$$

with

$$[A](z=0) \;=\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right)\,\left(\sum_a (aw)\,F_a(\tilde{x}) - \tilde{x}\right)^2. \tag{29}$$

4.2 The [D] Term

We now move on to the second line in the braces of Eq.(23), which corresponds to the $[D]$ term:

$$
\begin{aligned}
[D] \;&:=\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i)\right] \\
&\times \left\{ \sum_{i=1}^{N} \left\langle \left(\sum_a (aw)\ \langle F_a \rangle_{\Delta x}^{x_i(y)} - \sum_a (aw)\ \left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \right)^2 \right\rangle_y^{k_i} \right. \\
&\qquad \left. \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{k_i,\ \sum_{i=1}^{N} k_i} \right\}.
\end{aligned}
\tag{30}
$$

Notice that

$$
\sum_a (aw)\ \langle F_a \rangle_{\Delta x}^{x_i(y)} = \sum_a (aw)\ \langle F_a(x_{ij}) \rangle_{\Delta x_{ij}}^{x_i(y_i)} = \sum_a (aw)\ \langle F_a(\tilde{x} + y_i\ \tan\phi + \Delta x_{ij}) \rangle_{\Delta x_{ij}}^{x_i(y_i)}
$$

gives the average coordinate as obtained from a single primary cluster, which is labeled by i, since all the electrons from the primary cluster give the same coordinate after averaging over Δx. In the narrow pad limit, this average coordinate coincides exactly with the primary cluster position:

$$
\lim_{w/\sigma_{PR} \to 0} \sum_a (aw)\ \langle F_a(x_{ij}) \rangle_{\Delta x_{ij}}^{x_i(y_i)} \;=\; \langle x_{ij} \rangle_{\Delta x_{ij}}^{x_i(y_i)} \;=\; \tilde{x} + y_i\ \tan\phi,
$$

and its average over y_i coincides with the track position in the x direction:

$$
\lim_{w/\sigma_{PR} \to 0} \sum_a (aw)\ \left\langle \langle F_a(x_{ij}) \rangle_{\Delta x_{ij}}^{x_i(y_i)} \right\rangle_{y_i}^{k_i} \;=\; \left\langle \langle x_{ij} \rangle_{\Delta x_{ij}}^{x_i(y_i)} \right\rangle_{y_i}^{k_i} \;=\; \tilde{x}.
$$

Their difference, $y_i\ \tan\phi$, being squared and averaged in the bracket is the primary cluster position projected to the x-axis as measured from the track position $\tilde{x}$.

The $[D]$ term, therefore, represents the contribution to the x-resolution from the fluctuation of primary cluster positions along the track combined with the contribution from the fluctuation of the avalanche charge created by all the electrons from each cluster. When $\tan\phi = 0$, the fluctuation of the primary cluster positions along the track will not change their x-positions and hence the $[D]$ term will vanish.

In the narrow pad limit, we have

$$
\text{the 2nd term} \;\to\; \sum_{i=1}^{N} \tan^2\!\phi \left\langle (y_i)^2 \right\rangle_{y_i}^{k_i} \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{k_i,\ \sum_{i=1}^{N} k_i},
$$

where the variance, $\langle (y_i)^2 \rangle_{y_i}^{k_i}$, can be approximated by

$$
\left\langle (y_i)^2 \right\rangle_{y_i}^{k_i} \;\simeq\; \frac{L^2}{12}
$$

as long as $\sigma_d \ll L$. In this limit, the $[D]$ term becomes

$$\text{the } [D] \text{ term} \quad \rightarrow \quad \tan^2\phi \; \frac{L^2}{12} \; \left\langle \sum_{i=1}^{N} \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{k_i,\, \sum_{i=1}^{N} k_i} \right\rangle_{N,k} . \tag{31}$$

Comparison of this with Eq.(24) suggests that these have the same structure if you replace the role of individual avalanche charges (G_{ij}) caused by individual electrons by those of individual avalanche clusters ($\hat{G}_i := \sum_{j=1}^{k_i} G_{ij}$) caused by individual primary clusters, which fact prompts us to define $\hat{N}_{eff}$:

$$\begin{aligned} \hat{N}_{eff} \;\; := & \left[\left\langle \sum_{i=1}^{N} \left\langle \left(\sum_a (aw) \; \langle F_a \rangle_{\Delta x}^{x_i(y)} - \sum_a (aw) \; \left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \right)^2 \right\rangle_y^{k_i} \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{k_i,\, \sum_{i=1}^{N} k_i} \right\rangle_{N,k} \right]^{-1} \\ & \times \tan^2\phi \left(\frac{L^2}{12} \right), \end{aligned} \tag{32}$$

which reduces to

$$\hat{N}_{eff} \;\; \simeq \;\; \left[\left\langle \sum_{i=1}^{N} \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{k_i,\, \sum_{i=1}^{N} k_i} \right\rangle_{N,k} \right]^{-1} \tag{33}$$

in the narrow pad limit and as long as $\sigma_d \ll L$ therefore it can be regarded as the effective number of primary clusters for the angle term. To see this more clearly, let us make the following bold assumption:

$$\overline{\hat{G}} \;\; := \;\; \left\langle \sum_{j=1}^{k_i} G_{ij} \right\rangle \;\; \sim \;\; \frac{\sum_{i=1}^{N} \left(\sum_{j=1}^{k_i} G_{ij} \right)}{N} \;\; = \;\; \frac{\sum_{i=1}^{N} \hat{G}_i}{N},$$

which is by no means a good approximation in most practice (where N is not so large). We then have

$$\hat{N}_{eff} \;\; \sim \;\; \left[\left\langle \frac{1}{N} \right\rangle_N \left\langle \left(\frac{\hat{G}}{\overline{\hat{G}}} \right)^2 \right\rangle_{\hat{G}} \right]^{-1}, \tag{34}$$

which has exactly the same form as the definition of N_{eff}, where the role of individual electrons are replaced by that of individual primary clusters. This indicates that $\hat{N}_{eff}$ should be significantly smaller than N_{eff}. A sample calculation performed using Eq.(33) shows $\hat{N}_{eff} \simeq 5$ for Ar-rich gas with a pad row width of 6.3 mm.

In the declustering limit where the diffusion dominates the pad length ($\sigma_d \gg L$), we have

$$\left\langle (y_i)^2 \right\rangle_{y_i}^{k_i} \;\; \simeq \;\; \sigma_d^2 \quad \text{(long drift limit: } \sigma_d \gg L)$$

and k_i will be at most one, the $[D]$ term becomes

$$\text{the } [D] \text{ term} \quad \rightarrow \quad \tan^2\phi \; \sigma_d^2 \; \left\langle \sum_{i=1}^{N} k_i \left\langle \left(\frac{G_{i1}}{\sum_{i=1}^{N} k_i G_{i1}} \right)^2 \right\rangle_G^{\sum_{i=1}^{N} k_i} \right\rangle_{N,k}$$

$$\simeq \quad \tan^2\phi \; \sigma_d^2 \; \left\langle \frac{1}{\sum_{i=1}^N k_i} \right\rangle_{N,k} \left\langle \left(\frac{G}{\bar{\bar{G}}}\right)^2 \right\rangle_G .$$

This implies that $\hat{N}_{eff} \to 0$ in the declustering limit.

Notice that in usual choice of pad length σ_d/L hardly exceeds 0.2, and this declustering limit is never reached. Instead, we are usually closer to the $\sigma_d \ll L$ situation which we discussed above. We quote here again the formula in this limit:

$$[D] \;\; \simeq \;\; \frac{1}{\hat{N}_{eff}} \left(\frac{L^2 \tan^2 \phi}{12} \right)$$

with

$$\hat{N}_{eff} \;\; \simeq \;\; \left[\left\langle \sum_{i=1}^N \left\langle \left(\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle_G^{k_i,\, \sum_{i=1}^N k_i} \right\rangle_{N,k} \right]^{-1} , \tag{35}$$

which suggests that the angular pad effect is controlled by the projected track length ($=L\tan\phi$) to the x-axis and the effective number of primary clusters.

4.3 The [G] Term

The third and fourth lines in the braces of Eq.(23) correspond to the $[G]$ term:

$$\begin{aligned} [G] \;\; := \;\; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i) \right] \\ & \times \Bigg\{ \Bigg\langle \Bigg(\sum_{i=1}^N \sum_a (aw) \left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \\ & \qquad - \sum_{i=1}^N \sum_a (aw) \left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \left\langle \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right\rangle_G^{k_i,\, \sum_{i=1}^N k_i} \Bigg)^2 \Bigg\rangle_G \Bigg\} \\ := \;\; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i) \right] \\ & \times \Bigg\{ \Bigg\langle \Bigg(\sum_{i=1}^N \sum_a (aw) \left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \left[\frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} - \left\langle \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right\rangle_G^{k_i,\, \sum_{i=1}^N k_i} \right] \Bigg)^2 \Bigg\rangle_G \Bigg\} . \end{aligned}$$

This shows that the $[G]$ term is from gas gain fluctuation. Since the Δx and y averages have already been taken for primary clusters and electrons belonging to them, there must be only very weak dependence left on k_i for the double average defined in Eq.(21):

$$\left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \;\; := \;\; \int_{-\frac{\Delta Y}{2}}^{+\frac{\Delta Y}{2}} \frac{dy_i}{\Delta Y} \, \bar{\bar{P}}_{SI}(k_i, y_i) \, \langle F_a \rangle_{\Delta x}^{x_i(y_i)} .$$

Then the summation over i can be moved into the square brackets and then the numerators and the denominators of the first and the second terms in there cancel and consequently the contents of the square brackets vanish. For this reason, the $[G]$ term is negligible in practice as can be directly confirmed by sample calculations. Notice also that this term exactly vanishes if $\tan\phi = 0$, since the double average becomes truly independent of k_i in that case.

4.4 The [A] Term

The last line in the braces of Eq.(23) corresponds to the $[A]$ term:

$$\begin{aligned} [A] \;:=\; & \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \sum_{N=1}^{\infty} P_{PI}(N;\bar{N}) \prod_{i=1}^{N} \left[\sum_{k_i=0}^{\infty} \bar{\bar{P}}_{SI}(k_i) \right] \\ & \times \left\{ \left(\sum_a (aw) \sum_{i=1}^{N} \left\langle \langle F_a \rangle_{\Delta x}^{x_i(y)} \right\rangle_y^{k_i} \left\langle \frac{\sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right\rangle_G^{k_i,\, \sum_{i=1}^{N} k_i} - \tilde{x} \right)^2 \right\}. \end{aligned} \tag{36}$$

Here again, since the double average defined in Eq.(21) has a very weak dependence on k_i, we can move the summation over i into the bracket for the gain average and then the numerator and the denominator cancel there. The $[A]$ term is hence almost purely geometric, representing the well known hodoscope effect and the S-shape systematics, which are significant at short drift distances and for small $\tan\phi$ but quickly go to zero as σ_d/w or $(L/w)\tan\phi$ approaches 1.

5 Sample Calculations

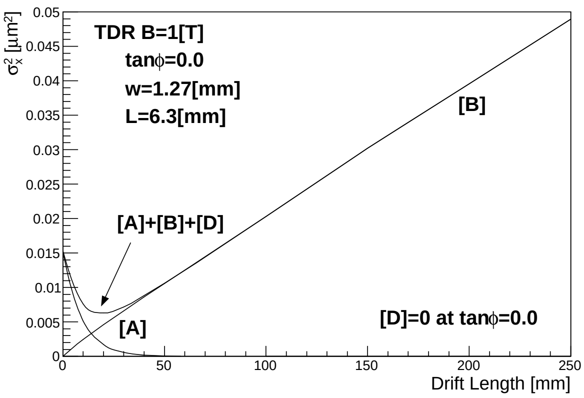

Figure 4: Spatial resolution as a function of drift distance for the TDR gas at 1T, $\tan\phi = 0.0$, $w = 1.27$mm and $L = 6.3$mm.

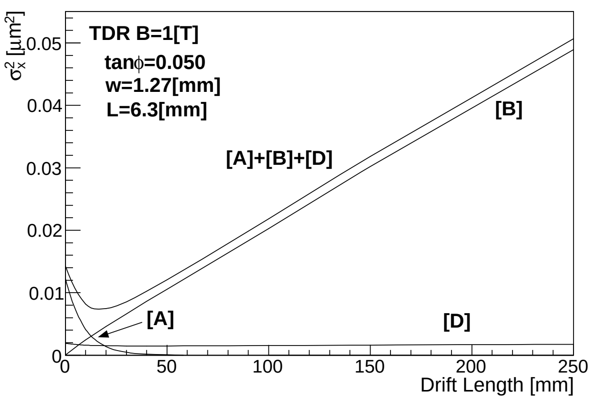

Figure 5: Spatial resolution as a function of drift distance for the TDR gas at 1T, $\tan\phi = 0.050$, $w = 1.27$mm and $L = 6.3$mm.

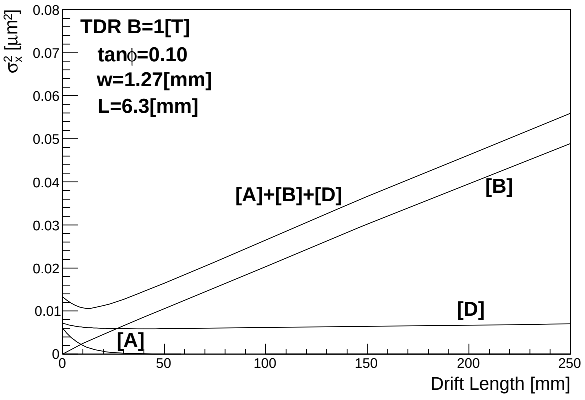

Figure 6: Spatial resolution as a function of drift distance for the TDR gas at 1T, $\tan\phi = 0.100$, $w = 1.27$mm and $L = 6.3$mm.

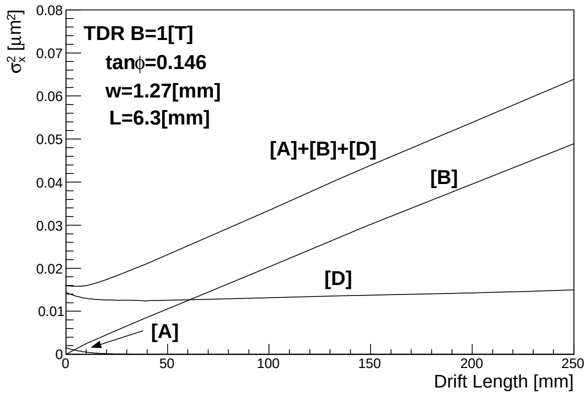

Figure 7: Spatial resolution as a function of drift distance for the TDR gas at 1T, $\tan\phi = 0.146$, $w = 1.27$mm and $L = 6.3$mm.

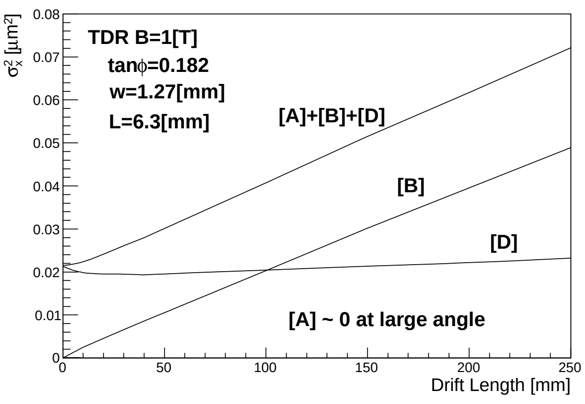

Figure 8: Spatial resolution as a function of drift distance for the TDR gas at 1T, $\tan\phi = 0.182$, $w = 1.27$mm and $L = 6.3$mm.

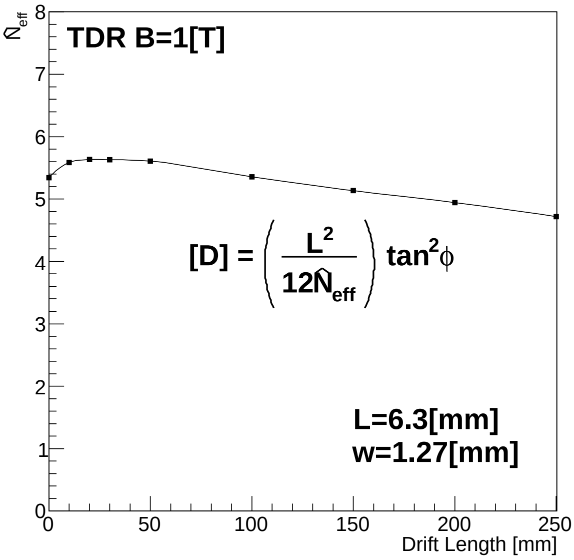

Figure 9: $\hat{N}_{eff}$ as a function of drift distance.

A Pad Response Function and Its Average

When the pad response function is a Gaussian with a standard deviation of σ_{PR}, $F_a(x)$ is given by

$$F_a(x;\sigma_{PR}) \;:=\; \int_{\left(a-\frac{1}{2}\right)w}^{\left(a+\frac{1}{2}\right)w} \frac{dx'}{\sqrt{2\pi}\sigma_{PR}} \exp\left[-\frac{1}{2}\left(\frac{x'-x}{\sigma_{PR}}\right)^2\right]. \tag{37}$$

$$\langle F_a\rangle_{\Delta x}^{x_i(y_i)} \;:=\; \int_{-\infty}^{+\infty} d\Delta x_{ij} P_D(\Delta x_{ij};\sigma_d)\, F_a(\tilde{x}+y_i\tan\phi+\Delta x_{ij};\sigma_{PR}) \tag{38}$$

It is then straightforward to calculate its diffusion average:

$$\begin{aligned}
\langle F_a\rangle_{\Delta x}^{x_i(y_i)} \;&:=\; \int_{-\infty}^{+\infty} d\Delta x P_D(\Delta x;\sigma_d)\, F_a(\tilde{x}+y_i\tan\phi+\Delta x) \\
&= \int_{\left(a-\frac{1}{2}\right)w}^{\left(a+\frac{1}{2}\right)w} \frac{dx'}{\sqrt{2\pi}\sigma_{PR}} \int_{-\infty}^{+\infty} \frac{d\Delta x}{\sqrt{2\pi}\sigma_d} \exp\left[-\frac{1}{2}\left\{\left(\frac{\Delta x}{\sigma_d}\right)^2+\left(\frac{\Delta x+\tilde{x}+y_i\tan\phi-x'}{\sigma_{PR}}\right)^2\right\}\right] \\
&= \int_{\left(a-\frac{1}{2}\right)w}^{\left(a+\frac{1}{2}\right)w} \frac{dx'}{\sqrt{2\pi}\sigma_{PR}} \int_{-\infty}^{+\infty} \frac{d\Delta x}{\sqrt{2\pi}\sigma_d} \exp\left[-\frac{1}{2}\left\{\left(\frac{1}{\sigma_d^2}+\frac{1}{\sigma_{PR}^2}\right)(\Delta x)^2\right.\right. \\
&\qquad\qquad \left.\left. -2\,\frac{x'-\tilde{x}-y_i\tan\phi}{\sigma_{PR}^2}\Delta x+\frac{(x'-\tilde{x}-y_i\tan\phi)^2}{\sigma_{PR}^2}\right\}\right] \\
&= \int_{\left(a-\frac{1}{2}\right)w}^{\left(a+\frac{1}{2}\right)w} \frac{dx'}{\sqrt{2\pi}\sigma_{PR}} \int_{-\infty}^{+\infty} \frac{d\Delta x}{\sqrt{2\pi}\sigma_d} \exp\left[-\frac{1}{2}\left\{\left(\frac{\sigma_d^2+\sigma_{PR}^2}{\sigma_d^2\sigma_{PR}^2}\right)\left(\Delta x-\frac{\sigma_d^2(x'-\tilde{x}-y_i\tan\phi)}{\sigma_d^2+\sigma_{PR}^2}\right)^2\right.\right. \\
&\qquad\qquad \left.\left. -\frac{\sigma_d^2(x'-\tilde{x}-y_i\tan\phi)^2}{\sigma_{PR}^2(\sigma_d^2+\sigma_{PR}^2)}+\frac{(x'-\tilde{x}-y_i\tan\phi)^2}{\sigma_{PR}^2}\right\}\right] \\
&= \int_{\left(a-\frac{1}{2}\right)w}^{\left(a+\frac{1}{2}\right)w} \frac{dx'}{\sqrt{2\pi}\sigma_{PR}} \int_{-\infty}^{+\infty} \frac{d\Delta x}{\sqrt{2\pi}\sigma_d} \exp\left[-\frac{1}{2}\left\{\left(\frac{\sigma_d^2+\sigma_{PR}^2}{\sigma_d^2\sigma_{PR}^2}\right)\left(\Delta x-\frac{\sigma_d^2(x'-\tilde{x}-y_i\tan\phi)}{\sigma_d^2+\sigma_{PR}^2}\right)^2\right.\right. \\
&\qquad\qquad \left.\left. +\frac{(x'-\tilde{x}-y_i\tan\phi)^2}{\sigma_d^2+\sigma_{PR}^2}\right\}\right] \\
&= \int_{\left(a-\frac{1}{2}\right)w}^{\left(a+\frac{1}{2}\right)w} \frac{dx'}{\sqrt{2\pi(\sigma_d^2+\sigma_{PR}^2)}} \exp\left[-\frac{1}{2}\left\{\frac{(x'-\tilde{x}-y_i\tan\phi)^2}{\sigma_d^2+\sigma_{PR}^2}\right\}\right],
\end{aligned}$$

indicating that

$$\langle F_a\rangle_{\Delta x}^{x_i(y_i)} \;=\; F_a\left(\tilde{x}+y_i\tan\phi;\sqrt{\sigma_{PR}^2+\sigma_d^2}\right). \tag{39}$$

B Some General Formulae

Theorem B.1

If $F_a(x)$ satisfies

$$F_a(x+w) \quad = \quad F_{a-1}(x)$$

and

$$\sum_a F_a(x) \quad = \quad 1$$

for $\forall x$, then we have

$$\int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+w\varepsilon+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+w\varepsilon+\Delta x)\rangle_{\Delta x}^{\tilde{x}+w\varepsilon} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}+w\varepsilon}$$

$$= \quad \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x)\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}} .$$

Proof

First notice that it suffices to prove

$$\int_{-1/2+\varepsilon}^{+1/2+\varepsilon} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x)\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}}$$

$$= \quad \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x)\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}} ,$$

by change of variables ($\tilde{x} \to \tilde{x} + w\varepsilon$). Differentiating the left-hand side by ε, we get

$$\frac{d}{d\varepsilon} \int_{-1/2+\varepsilon}^{+1/2+\varepsilon} d\left(\frac{\tilde{x}}{w}\right) \left\langle \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x)\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}}$$

$$= \quad \left\langle \frac{d}{d\varepsilon} \int_{-1/2+\varepsilon}^{+1/2+\varepsilon} d\left(\frac{\tilde{x}}{w}\right) \left(\sum_a (aw)\, F_a(\tilde{x}+\Delta x) - \sum_a (aw)\ \langle F_a(\tilde{x}+\Delta x)\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}}$$

$$= \quad \left\langle \left(\sum_a (aw)\, F_a(w/2+w\varepsilon+\Delta x) - \left\langle \sum_a (aw)\, F_a(w/2+w\varepsilon+\Delta x) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right.$$

$$\left. - \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon+\Delta x) - \left\langle \sum_a (aw)\, F_a(-w/2+w\varepsilon+\Delta x) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}}$$

$$= \quad \left\langle \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon+\Delta x+w) - \left\langle \sum_a (aw)\, F_a(-w/2+w\varepsilon+\Delta x+w) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right.$$

$$- \left(\sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) - \left\langle \sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \Bigg\rangle_{\Delta x}^{\tilde{x}}$$

$$= \left\langle \left(\sum_a (aw)\, F_{a-1}(-w/2 + w\varepsilon + \Delta x) - \left\langle \sum_a (aw)\, F_{a-1}(-w/2 + w\varepsilon + \Delta x) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right.$$

$$\left. - \left(\sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) - \left\langle \sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}}$$

$$= \left\langle \left(\sum_a (aw + w)\, F_a(-w/2 + w\varepsilon + \Delta x) - \left\langle \sum_a (aw + w)\, F_a(-w/2 + w\varepsilon + \Delta x) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right.$$

$$\left. - \left(\sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) - \left\langle \sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}}$$

$$= \left\langle \left(\sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) - \left\langle \sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right.$$

$$\left. - \left(\sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) - \left\langle \sum_a (aw)\, F_a(-w/2 + w\varepsilon + \Delta x) \right\rangle_{\Delta x}^{\tilde{x}} \right)^2 \right\rangle_{\Delta x}^{\tilde{x}} = 0.$$

The left-hand side is hence independent of ε and can be set to 0. [Q.E.D.]

Theorem B.2

If $F_a(x)$ satisfies

$$F_a(x + w) = F_{a-1}(x)$$

and

$$\sum_a F_a(x) = 1$$

then for $\forall \varepsilon$ we have

$$\int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left(\sum_a (aw)\, F_a(\tilde{x} + w\varepsilon) - (\tilde{x} + w\varepsilon) \right) = 0$$

Proof

First notice that it suffices to prove

$$\int_{-1/2+\varepsilon}^{+1/2+\varepsilon} d\left(\frac{\tilde{x}}{w}\right) \left(\sum_a (aw)\, F_a(\tilde{x}) - (\tilde{x}) \right) = 0$$

by change of variables ($\tilde{x} \to \tilde{x} + w\varepsilon$). Differentiating the left-hand side by ε, we get

$$\frac{d}{d\varepsilon} \int_{-1/2+\varepsilon}^{+1/2+\varepsilon} d\left(\frac{\tilde{x}}{w}\right) \left(\sum_a (aw)\, F_a(\tilde{x}) - (\tilde{x})\right)$$

$$= \left(\sum_a (aw)\, F_a(w/2+\varepsilon) - (w/2+\varepsilon)\right) - \left(\sum_a (aw)\, F_a(-w/2+\varepsilon) - (-w/2+\varepsilon)\right)$$

$$= \left(\sum_a (aw)\, F_a(-w/2+\varepsilon+w) - (-w/2+\varepsilon+w)\right) - \left(\sum_a (aw)\, F_a(-w/2+\varepsilon) - (-w/2+\varepsilon)\right)$$

$$= \left(\sum_a (aw)\, F_{a-1}(-w/2+\varepsilon) - (-w/2+\varepsilon+w)\right) - \left(\sum_a (aw)\, F_a(-w/2+\varepsilon) - (-w/2+\varepsilon)\right)$$

$$= \left(\sum_a (aw+w)\, F_a(-w/2+\varepsilon) - (-w/2+\varepsilon+w)\right) - \left(\sum_a (aw)\, F_a(-w/2+\varepsilon) - (-w/2+\varepsilon)\right)$$

$$= \sum_a w\, F_a(-w/2+\varepsilon) - w \;=\; 0. \qquad \text{[Q.E.D.]}$$

Theorem B.3

If $F_a(x)$ satisfies

$$F_a(x+w) \;=\; F_{a-1}(x)$$

and

$$\sum_a F_a(x) \;=\; 1$$

then for $\forall\varepsilon$ we have

$$\int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left(\sum_a (aw)\, F_a(\tilde{x}+w\varepsilon) - (\tilde{x}+w\varepsilon)\right)^2 \;=\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left(\sum_a (aw)\, F_a(\tilde{x}) - \tilde{x}\right)^2.$$

Proof

First notice that it suffices to prove

$$\int_{-1/2+\varepsilon}^{+1/2+\varepsilon} d\left(\frac{\tilde{x}}{w}\right) \left(\sum_a (aw)\, F_a(\tilde{x}) - (\tilde{x})\right)^2 \;=\; \int_{-1/2}^{+1/2} d\left(\frac{\tilde{x}}{w}\right) \left(\sum_a (aw)\, F_a(\tilde{x}) - \tilde{x}\right)^2.$$

by change of variables ($\tilde{x} \to \tilde{x} + w\varepsilon$). Differentiating the left-hand side by ε, we get

$$\frac{d}{d\varepsilon} \int_{-1/2+\varepsilon}^{+1/2+\varepsilon} d\left(\frac{\tilde{x}}{w}\right) \left(\sum_a (aw)\, F_a(\tilde{x}) - \tilde{x}\right)^2$$

$$= \left(\sum_a (aw)\, F_a(w/2+w\varepsilon) - (w/2+w\varepsilon)\right)^2 - \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon) - (-w/2+w\varepsilon)\right)^2$$

$$
\begin{aligned}
&= \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon+w) - (-w/2+w\varepsilon+w)\right)^2 \\
&\qquad\qquad\qquad - \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon) - (-w/2+w\varepsilon)\right)^2 \\
&= \left(\sum_a (aw)\, F_{a-1}(-w/2+w\varepsilon) - (-w/2+w\varepsilon+w)\right)^2 \\
&\qquad\qquad\qquad - \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon) - (-w/2+w\varepsilon)\right)^2 \\
&= \left(\sum_a (aw+w)\, F_a(-w/2+w\varepsilon) - (-w/2+w\varepsilon+w)\right)^2 \\
&\qquad\qquad\qquad - \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon) - (-w/2+w\varepsilon)\right)^2 \\
&= \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon) - (-w/2+w\varepsilon) + w\sum_a F_a(-w/2+w\varepsilon) - w\right)^2 \\
&\qquad\qquad\qquad - \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon) - (-w/2+w\varepsilon)\right)^2 \\
&= \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon) - (-w/2+w\varepsilon)\right)^2 \\
&\qquad\qquad - \left(\sum_a (aw)\, F_a(-w/2+w\varepsilon) - (-w/2+w\varepsilon)\right)^2 \;=\; 0
\end{aligned}
$$

The left-hand side is independent of ε and hence allows us to set $\varepsilon = 0$. [Q.E.D.]