Basic Physics Behind Operation of TPC

Part II

-- Application to MPGD Readout TPCs --

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Coordinate Measurements

Charge Centroid Method with Readout Pads

Fundamental Limits on Spatial Resolution
for a MPGD Readout TPC

Analytic Formulation of Spatial Resolution
Important Outcome from KEK Beam Tests
(Asia, Europe, and North America)

MPGD Readout TPC

Coordinate Measurement Process

Coordinate System

We set our coordinate system in such a way that the readout pads are arranged in a row to measure the x-coordinate with charge centroid method, the y-coordinate from the pad row number, and the z-coordinate from the drift time.

Basic Assumptions

For simplicity, we will consider for a while a charged particle at normal incidence. We also assume that the effect of delta-rays is negligible (good approximation if there is a strong enough B-field) so that all the track electrons can be regarded as starting from a single point when projected to the (x, z) plane. These track electrons drift towards the amplification region while experiencing diffusion. The track electrons are then gas amplified while experiencing further diffusion. As we have discussed, when we readout pad signals with a slow enough electronics, only the real charge arriving at a readout pad counts. The spatial width of the signal is then determined by the width of the real charge distribution on the pad plane as determined by the diffusion in the drift and the amplification regions. Notice that we are dealing with normal incidence for which angular pad effect is absent.

In what follows we start from an ideal situation with a perfect readout plane, switching on one-by-one complications expected for more realistic situations.

Fundamental Processes

Beam

Ionizations

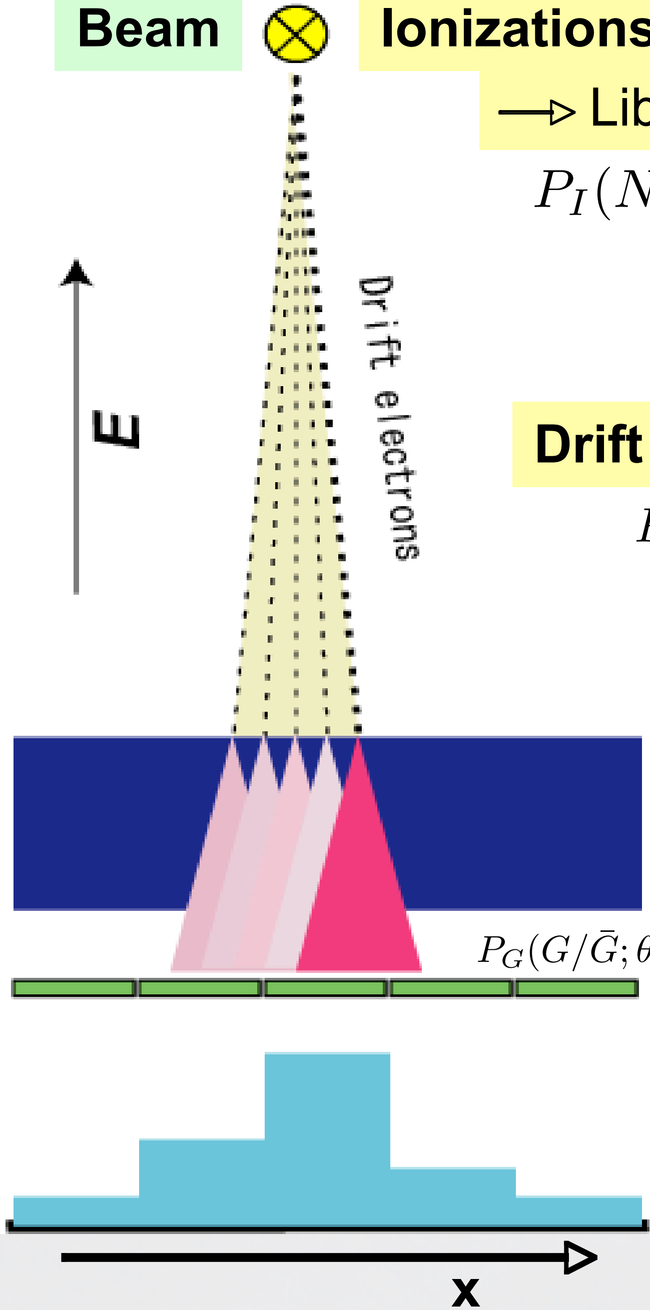
→ Liberation of Electrons

$P_I(N; \bar{N})$

Normal incidence
(no angle effect)

No δ-ray

Drift Volume



Drift and Diffusion

$$P_D(x_i; \sigma_d) = \frac{1}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{x_i^2}{2\sigma_d^2}\right)$$

$$\sigma_d = C_d\sqrt{z}$$

Amplification Gap

Amplification and further Diffusion

$$P_G(G/\bar{G}; \theta) = \frac{(\theta+1)^{\theta+1}}{\Gamma(\theta+1)} \left(\frac{G}{\bar{G}}\right)^{\theta} \exp\left(-(\theta+1)\left(\frac{G}{\bar{G}}\right)\right)$$

Readout Pads

Pad Response

→ **Coordinate**

Ionization Statistics

Ideal Readout Plane: Coordinate = Simple C.O.G.

PDF for C.O.G. of N electrons

We assume here an ideal readout plane that can measure the x-coordinates of individual track electrons exactly. The probability distribution function for the center of gravity of N track electrons is given by

$$P(\bar{x}) = \sum_{N=1}^{\infty} P_I(N;\bar{N}) \prod_{i=1}^{N} \left(\int dx_i \, P_D(x_i;\sigma_d) \right) \delta\left(\bar{x} - \frac{1}{N} \sum_{i=1}^{N} x_i \right)$$

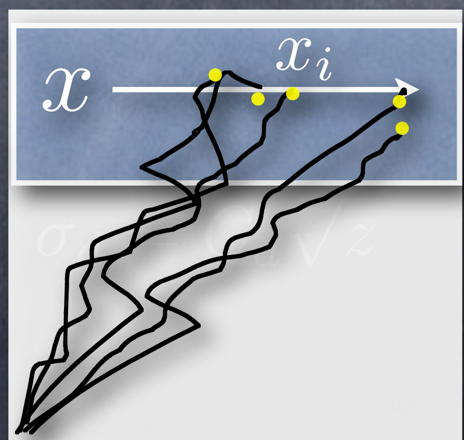
Ionization statistics

Ideal readout plane

Gaussian diffusion

$$P_D(x_i;\sigma_d) = \frac{1}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{x_i^2}{2\sigma_d^2} \right)$$

$$\sigma_d = C_d \sqrt{z}$$



where Cd is the diffusion coefficient and z is the drift length. The track is assumed to passed through the TPC at x=0 parallel with the readout plane and perpendicular to the pad rows.

The center of gravity of the N electrons is the best possible estimator of the incident x-coordinate of the track

$$\langle \bar{x} \rangle := \int d\bar{x} \, P(\bar{x}) \, \bar{x} = 0$$

The variance of the C.O.G. is then given by

$$\sigma_{\bar{x}}^2 := \int d\bar{x} \, P(\bar{x}) \, \bar{x}^2 = \sigma_d^2 \left\langle \frac{1}{N} \right\rangle =: \sigma_d^2 \frac{1}{N_{\mathrm{eff}}}$$

by definition. This leads us to

$$N_{\mathrm{eff}} := \frac{1}{\langle 1/N \rangle} < \langle N \rangle$$

What decides the spatial resolution is not the average number of ionization electrons but the inverse of the average of its inverse.

Gas Gain Fluctuation

Coordinate = Gain-Weighted Mean

PDF for gain-weighted mean

We now switch on the gas gain fluctuation and assume that the coordinate measured by the readout plane is the gain-weighted mean of the N ionization electrons.

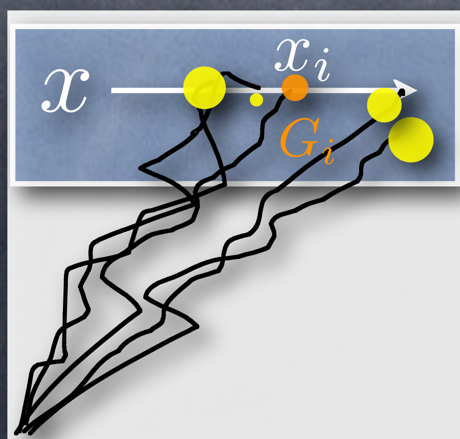
$$P(\bar{x}) = \sum_{N=1}^{\infty} P_I(N;\bar{N}) \prod_{i=1}^{N} \left[\int dx_i \, P_D(x_i;\sigma_d) \right.$$
$$\left. \times \int d\left(\frac{G_i}{\bar{G}}\right) P_G\left(\frac{G_i}{\bar{G}};\theta_{\rm pol}\right) \right] \delta\left(\bar{x} - \frac{\sum_{i=1}^{N} G_i \, x_i}{\sum_{i=1}^{N} G_i}\right)$$

Gas gain fluctuation

Gain-weighted mean

We used the Polya parameter as an index even though the PG is non-Polya in general.
Notice that

$$\sum_{i=1}^{N} G_i \approx N \, \bar{G}$$



Again we assume that the charged particle passed through the TPC at x=0 parallel with the readout plane and perpendicular to the pad rows.
The average of the gain-weighted mean has then no bias

$$\langle \bar{x} \rangle := \int d\bar{x} \, P(\bar{x}) \, \bar{x} = 0$$

The variance of the C.O.G. is then given by

$$\sigma_{\bar{x}}^2 := \int d\bar{x} \, P(\bar{x}) \, \bar{x}^2 \approx \sigma_d^2 \left\langle \frac{1}{N} \right\rangle \left\langle \left(\frac{G}{\bar{G}}\right)^2 \right\rangle =: \sigma_d^2 \frac{1}{N_{\rm eff}}$$

where use has been made of

$$\sum_{i=1}^{N} G_i \approx N \, \bar{G}$$

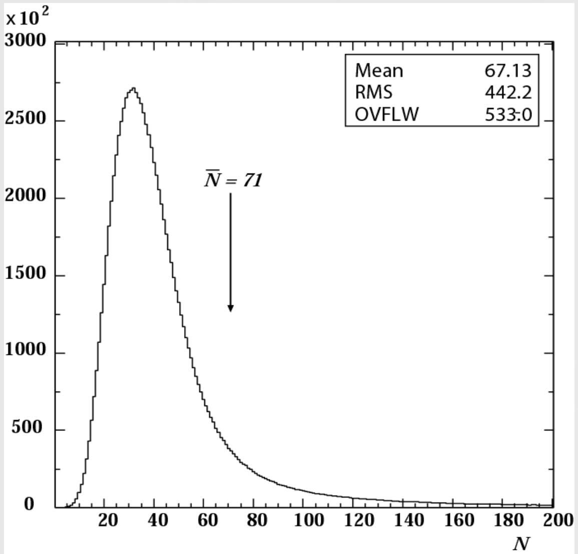
We hence have

$$N_{\rm eff} := \left[\left\langle \frac{1}{N} \right\rangle \left\langle \left(\frac{G}{\bar{G}}\right)^2 \right\rangle \right]^{-1} = \frac{1}{\langle 1/N \rangle} \left(\frac{1+\theta_{\rm pol}}{2+\theta_{\rm pol}}\right) < \langle N \rangle$$

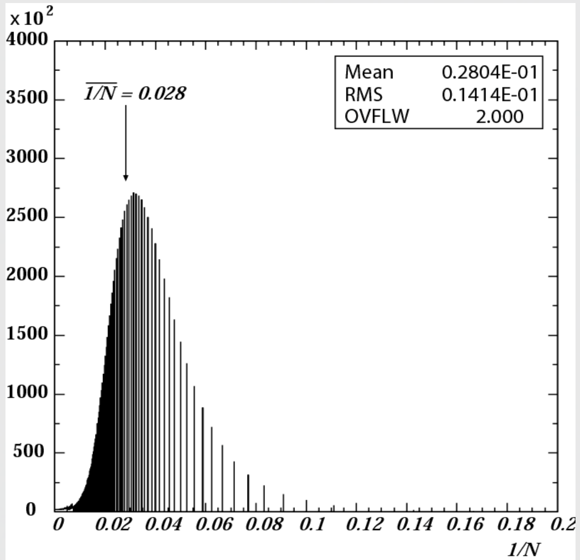
The gas gain fluctuation therefore further reduces the effective number of electrons.

Sample Calc. for Neff

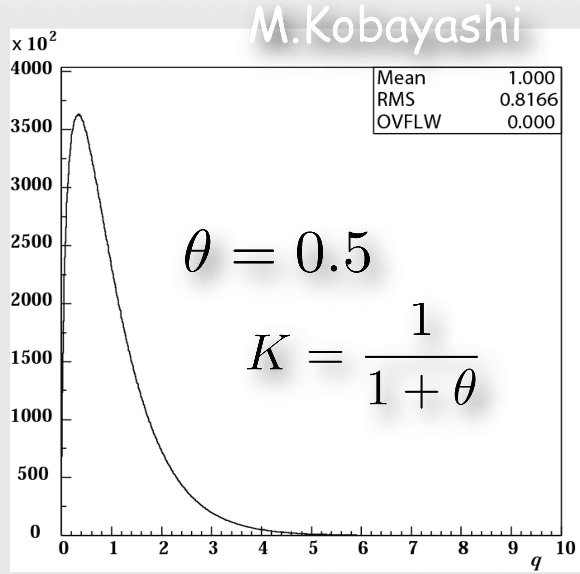
For 4 GeV pion and pad row pitch of 6mm in pure Ar



Distribution of N
($\langle N \rangle$ = 71)



Distribution of $1/N$
($\langle 1/N \rangle$ = 0.028)



Distribution of Q
(K = 0.67)

In the case of Snyder's model, gain fluctuation is exponential and K=1 (theta=0) and the Neff is reduced by a factor of 2 by it. In the case of Legler's model, theta>0 and the reduction is less sever. If we assume theta=0.5, for instance, we have a factor of 1.5 reduction:

$$\left\langle \left(\frac{G}{\bar{G}} \right)^2 \right\rangle = 1 + \left(\frac{\sigma_G}{\bar{G}} \right)^2 \equiv 1 + K$$

$$N_{eff} = \left[\left\langle \frac{1}{N} \right\rangle \left\langle \left(\frac{G}{\bar{G}} \right)^2 \right\rangle \right]^{-1} = 21 < \langle N \rangle = 71$$

Finite Size Pads

Coordinate = Charge Centroid

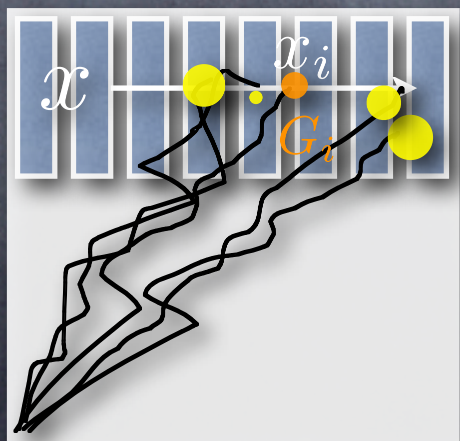
PDF for charge centroid

We now replace the continuous readout plane with an array of finite size pads. The finite size pads break the translational symmetry. We hence need to specify the track position relative to the pad center. The arrival point of i-th ionization electron is given by

diffusion : $\langle (\Delta x_i)^2 \rangle = \sigma_d^2 = C_d^2\, z$

$$x_i = \tilde{x} + \Delta x_i$$

track position



The charge in units of electron charge on a-th pad is given by

$$Q_a = \sum_{i=1}^{N} G_i\, F_a(\tilde{x} + \Delta x_i) + \Delta Q_a$$

where F_a is the normalized pad response function for a-th pad

$$\sum_a F_a(\tilde{x} + \Delta x_i) = 1$$

G_i is the gas gain for the i-th ionization electron, and ΔQ_a is the electronic noise:

$$\langle (\Delta Q_a)^2 \rangle = \sigma_E^2$$

The charge centroid is then given by

$$\bar{x} = \sum_a Q_a\,(a\,w) \Big/ \sum_a Q_a$$

with w being the pad pitch. The probability distribution function for charge centroid is

$$P(\bar{x}; \tilde{x}) = \sum_{i=1}^{\infty} P_I(N; \bar{N}) \prod_{i=1}^{N} \left[\int d\Delta x_i\, P_D(\Delta x_i; \sigma_d) \int d\left(\frac{G_i}{\bar{G}}\right) P_G\left(\frac{G_i}{\bar{G}}; \theta_{\rm pol}\right)\right]$$
$$\times \prod_a \left[\int d\Delta Q_a\, P_E(\Delta Q_a; \sigma_E) \int dQ_a\, \delta\left(Q_a - \sum_{i=1}^{N} G_i\, F_a(\tilde{x} + \Delta x_i) - \Delta Q_a\right)\right]$$
$$\times\, \delta\left(\bar{x} - \frac{\sum_a Q_a\,(a\,w)}{\sum_a Q_a}\right)$$

Variance of charge centroid

In order to take into account the effect of finite size pads as known as the S-shape systematics, we define the variance by

$$\sigma^2_{\bar{x}} := \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\left(\frac{\tilde{x}}{w}\right) \int d\bar{x}\, P(\bar{x};\tilde{x})\, (\bar{x}-\tilde{x})^2$$

Substituting the PDF given above in this and with some arithmetics, we obtain

$$\sigma^2_{\bar{x}} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\left(\frac{\tilde{x}}{w}\right) \left[[A] + \frac{1}{N_{\rm eff}}[B]\right] + [C]$$

where

$$[A] := \left(\sum_a (a\,w)\,\langle F_a(\tilde{x}+\Delta x)\rangle - \tilde{x}\right)^2$$

is a purely geometric term corresponding to the S-shape systematics due to the finite pad pitch and disappears rapidly as z increases. On the other hand,

$$[B] := \sum_{a,b} a\,b\,w^2\,\langle F_a(\tilde{x}+\Delta x) F_b(\tilde{x}+\Delta x)\rangle - \left(\sum_a a\,w\,\langle F_a(\tilde{x}+\Delta x)\rangle\right)^2$$

is a term representing the contributions from diffusion, gas gain fluctuation, and finite pad pitch. The contribution of this term scales as 1/Neff and dominates the spatial resolution at a long drift distance. The last term

$$[C] := \left(\frac{\sigma_E}{\bar{G}}\right)^2 \left\langle\frac{1}{N^2}\right\rangle \sum_a (a\,w)^2$$

is an electronic noise term, which is z-independent and scales as $\langle 1/N^2\rangle$.

The correlation function and the averaged pad response functions are defined by

$$\langle F_a(\tilde{x}+\Delta x) F_b(\tilde{x}+\Delta x)\rangle := \int d\Delta x\, P_D(\Delta x;\sigma_d)\, F_a(\tilde{x}+\Delta x)\, F_b(\tilde{x}+\Delta x)$$

and

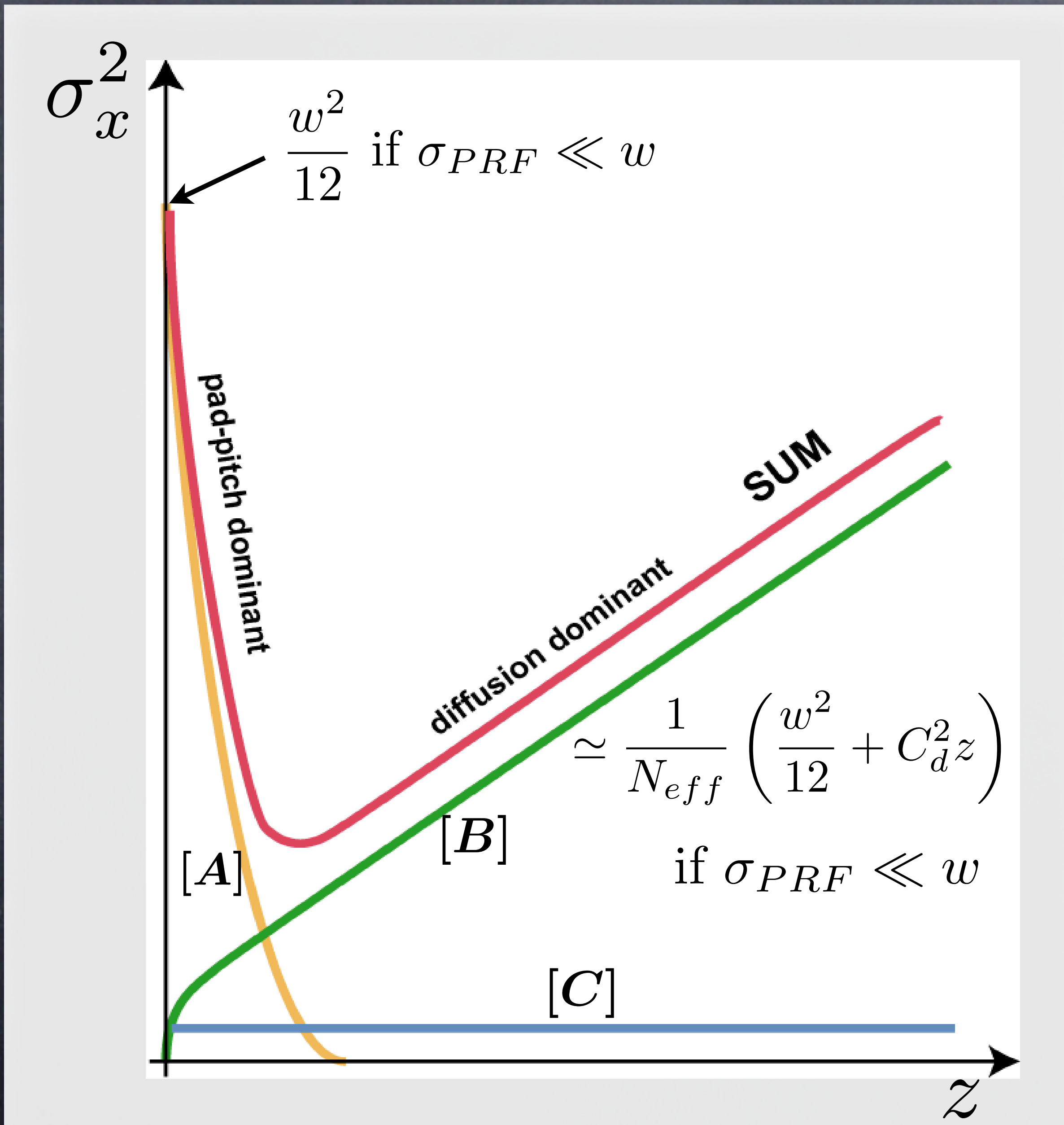
$$\langle F_a(\tilde{x}+\Delta x)\rangle := \int d\Delta x\, P_D(\Delta x;\sigma_d)\, F_a(\tilde{x}+\Delta x)$$

For a delta-function like pad response fun. we have an asymptotic form at large z of

$$\sigma^2_{\bar{x}} \simeq \frac{1}{N_{\rm eff}}\left(\frac{w^2}{12} + C_d^2\, z\right)$$

if electronic noise is negligible.

Interpretation



[A] Purely geometric term (S-shape systematics from finite pad pitch): rapidly disappears as Z increases

[B] Diffusion, gas gain fluctuation & finite pad pitch term: scales as $1/N_{eff}$, for delta-fun like PRF asymptotically:

$$\sigma_{\bar{x}}^2 \simeq \frac{1}{N_{eff}} \left(\frac{w^2}{12} + C_d^2 z \right)$$

[C] Electronic noise term: Z-independent, scales as $\langle 1/N^2 \rangle$

Application to Micromegas

$(0, 1/\sqrt{12})$: hodoscope limit

- For a delta-function like PRF, there is a scaling law: σ_x/w depends only on σ_d/w and N_{eff}
- The formula has a fixed point

$$(0, 1/\sqrt{12})$$

- Full formula enters asymptotic region at

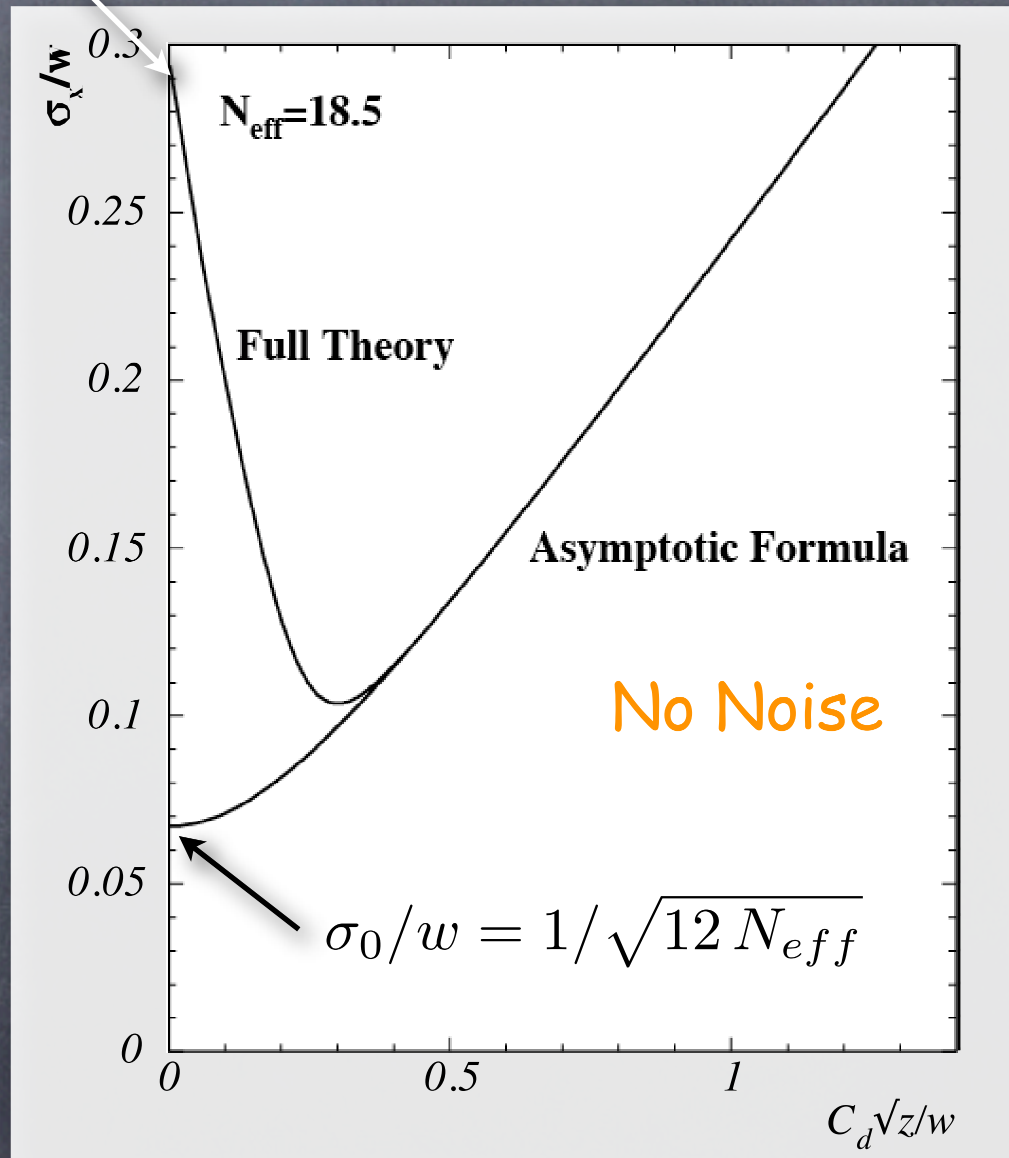
$$\sigma_d/w \simeq 0.4$$

- Full formula has a minimum of

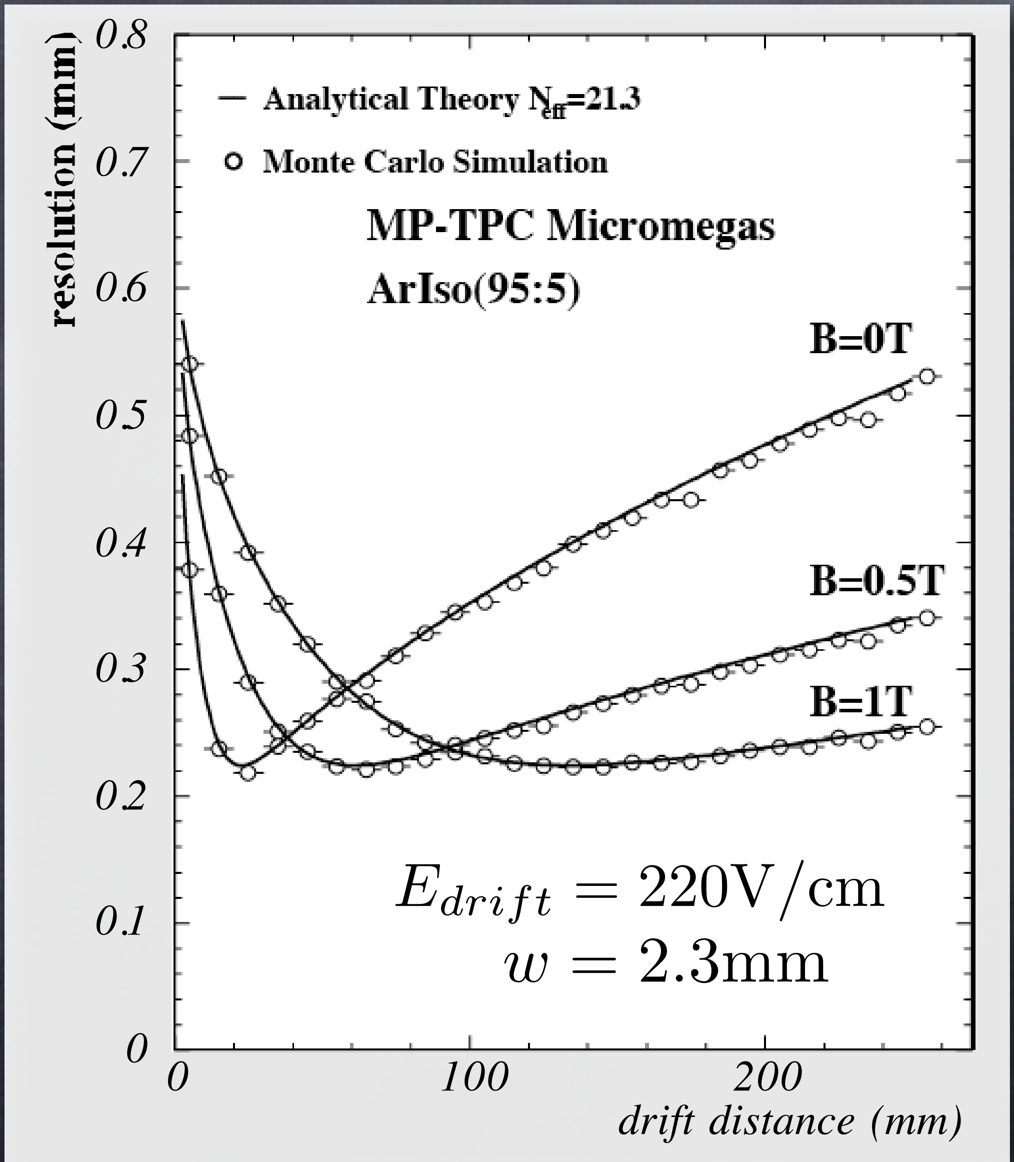
at

$$\sigma_x/w \simeq 0.1$$

$$\sigma_d/w \simeq 0.3$$



Comparison with MC



- Theory reproduces the Monte Carlo simulation very well !
- We can estimate the resolution analytically

drift distance

$$\sigma_x = \sigma_x(z; w, C_d, N_{eff}, [f_j])$$

pad pitch

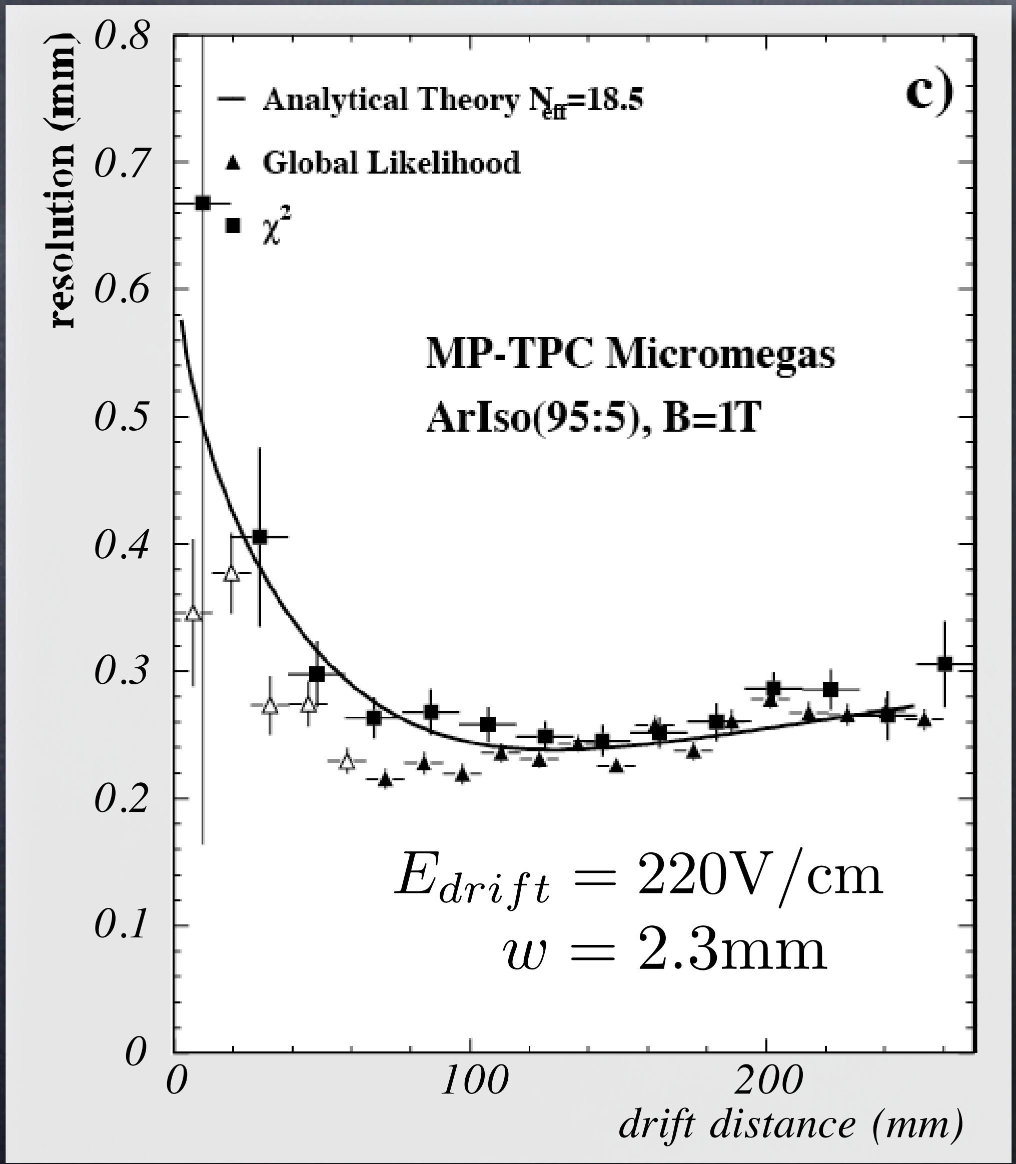
diffusion const.

pad response function

δ-fun. for MM: $\sigma_{PRF} \simeq 12\mu\mathrm{m}$

gauss. for GEM: $\sigma_{PRF} \simeq 350\mu\mathrm{m}$

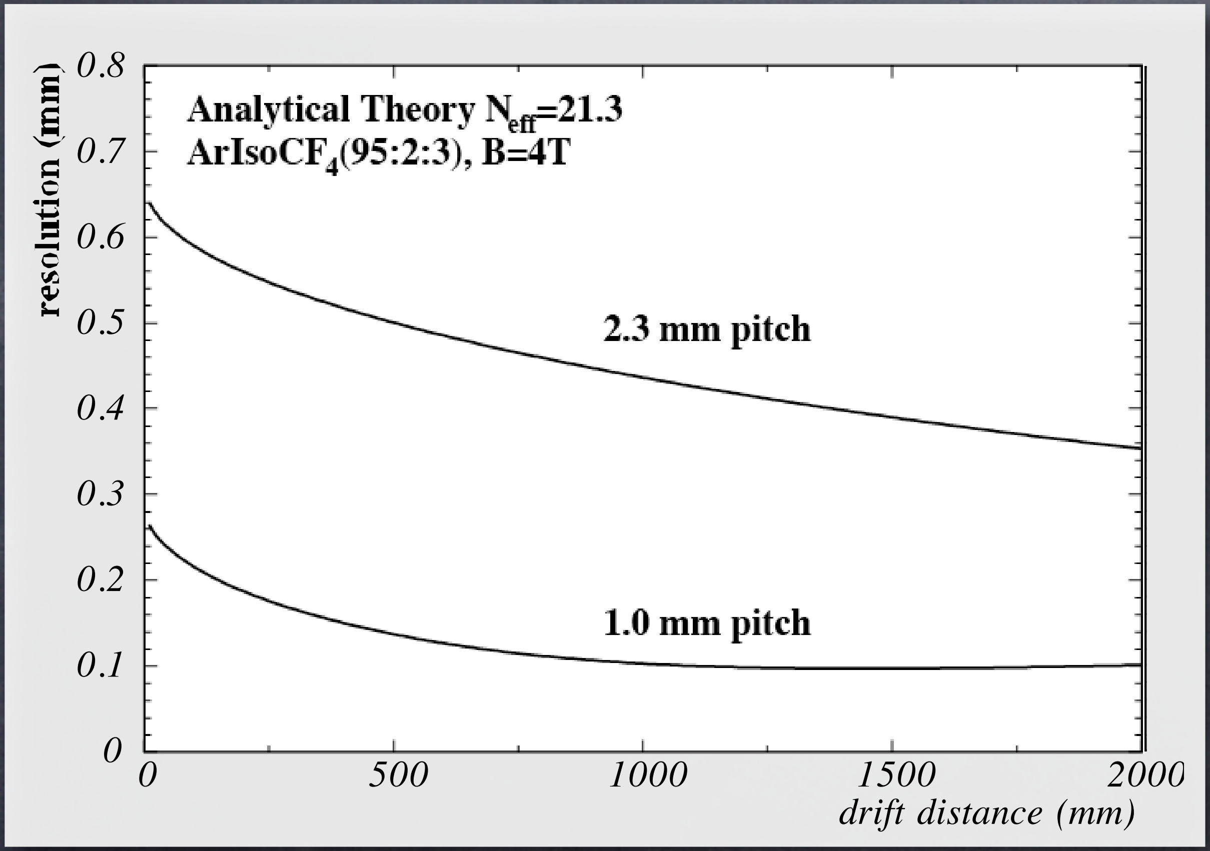
Comparison with Measurements



KEK beam test data

- Theory reproduces the data well.
- Underestimation in the data of σ_x at short drift distance is due to track bias caused by S-shape systematics.
- Global likelihood method eliminates the S-shape systematics at short distance when possible and hence gives better resolution than the simple charge centroid method used in the chi-square fit.

Extrapolation to LC TPC



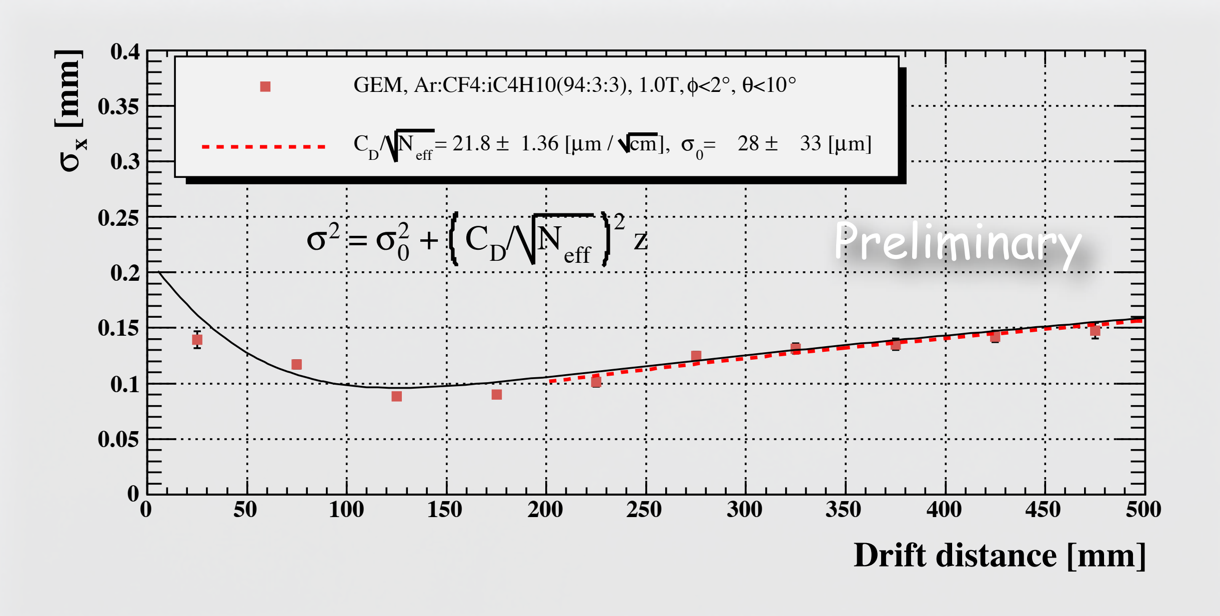
- Need to reduce pad size relative to PRF
  - Resistive anode for MM.
  - Digital pixel readout for MM corresponding to an ideal readout plane to avoid the effect of gain fluctuation (the best if feasible).
  - Defocusing + narrow (1mm) pad for GEM.

Recent results seem promising for both resistive anode and digital pixel readout schemes (Paul's talk)!

Application to GEM

TU-TPC test at KEK cryo hall (Dec. 2007)

- In the case of GEM, there is no simple scaling as with micromegas, since there is an additional dimensionful parameter that is the intrinsic signal width ($\sigma_{\rm PRF}$) which is determined by the diffusion in the transfer and induction gaps. When it is large enough compared with the pad pitch we can avoid the hodoscope effect at a short drift distance.



The theory assumes

$$v_{drift} = 4.2\,[\mathrm{cm}/\mu\mathrm{s}]$$

from drift time data and

$$\begin{aligned}\sigma_{PRF}^2 &= \sigma_{PR}(0)^2 - \frac{w^2}{12} \\ &= (270\,[\mu\mathrm{m}])^2\end{aligned}$$

$$C_D = 128\,[\mu\mathrm{m}/\sqrt{\mathrm{cm}}]$$

from charge width data and

$$N_{eff} = 22 \times (10./6.3) = 35$$

from MP-TPC result.

The TU-TPC data indicates

$$N_{eff} = (128/22) = 34 \pm 4$$

in good agreement with the MP-TPC result.

How to Measure Cd?

A detour

The average charge on a-th pad

The average charge on a-th pad is given by

$$\langle Q_a(\tilde{x})\rangle = NG\,\langle F_a(\tilde{x}+\Delta x)\rangle$$

resulting in the average charge fraction

$$\begin{aligned}
\langle Q_a(\tilde{x})\rangle/(\bar{N}\bar{G}) &= \langle F_a(\tilde{x}+\Delta x)\rangle \\
&:= \int d\Delta x\, P_D(\Delta x;\sigma_d)\, F_a(\tilde{x}+\Delta x) \\
&= \int_{a\,w-\tilde{x}-w/2}^{a\,w-\tilde{x}+w/2} d\Delta x\, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{\Delta x}{\sigma}\right)^2\right] \\
&= \int_{-w/2}^{+w/2} d\xi\, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{\hat{x}+\xi}{\sigma}\right)^2\right]
\end{aligned}$$

where

$$\hat{x} := a\,w - \tilde{x}$$

is the location of the pad center measured from the incident position of the track and

$$\sigma^2 := \sigma_{\rm PRF}^2 + \sigma_d^2 = \sigma_{\rm PRF}^2 + C_d^2\, z$$

is the squared sum of the intrinsic width of the pad response function at z=0 and the width due to diffusion in the drift region. We can hence define a normalized apparent pad response function

$$Q_{\rm PR}(\hat{x}) := \frac{1}{w}\int_{-w/2}^{+w/2} d\xi\, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{\hat{x}+\xi}{\sigma}\right)^2\right]$$

which has the variance

$$\begin{aligned}
\sigma_{\rm PR}^2 &= \int d\hat{x}\, Q_{\rm PR}(\hat{x})\,\hat{x}^2 \\
&= \frac{1}{w}\int_{-w/2}^{+w/2} d\xi \int_{-\infty}^{+\infty} d\hat{x}\, \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{\hat{x}+\xi}{\sigma}\right)^2\right]\hat{x}^2 \\
&= \frac{1}{w}\int_{-w/2}^{+w/2} d\xi\,(\sigma^2+\xi^2) \;=\; \sigma^2 + \frac{w^2}{12}
\end{aligned}$$

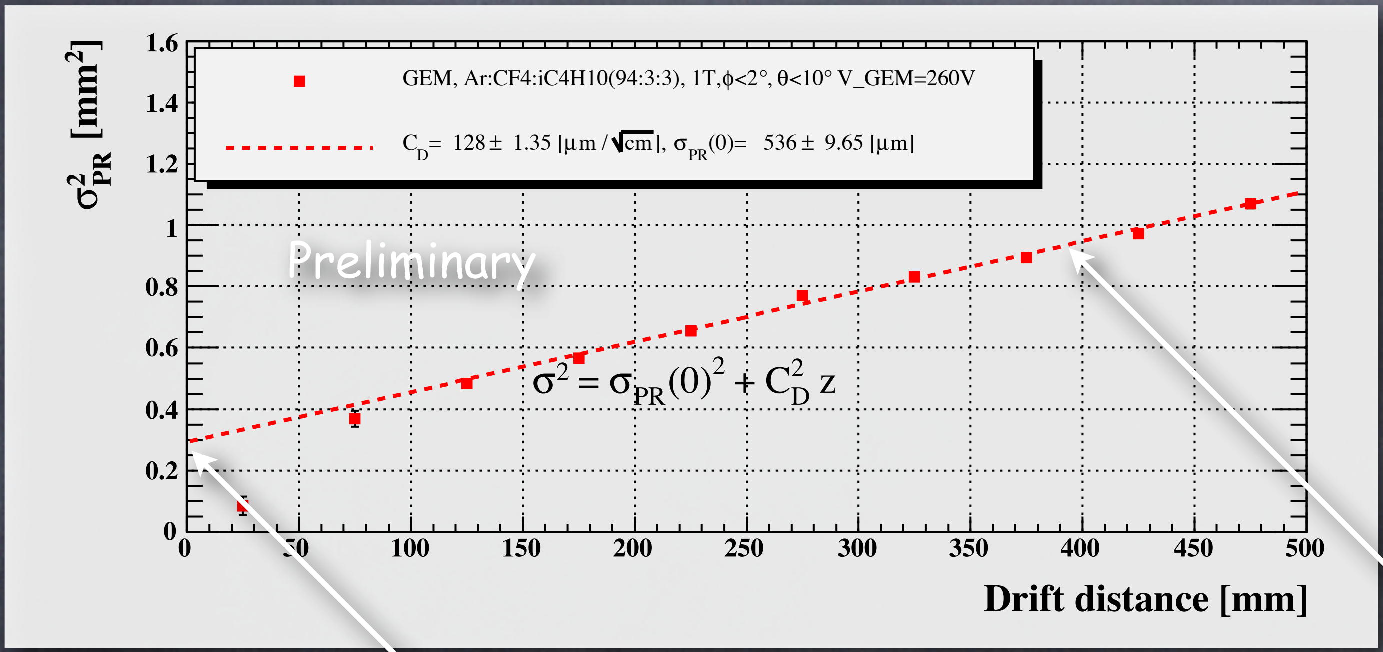
From this we obtain

$$\sigma_{\rm PR}^2(0) := \sigma_{\rm PR}^2 - C_d^2 z = \sigma_{\rm PRF}^2 + \frac{w^2}{12}$$

By plotting $\sigma_{\rm PR}^2$ as a function of z, we can hence extract Cd from the slope and $\sigma_{\rm PRF}^2$ from the intercept with the finite pad pitch correction of $w^2/12$.

Cd Measurement

TU-TPC test at KEK cryo hall (Dec. 2007)



Assuming

$$v_{drift} = 4.2\,[\mathrm{cm}/\mu\mathrm{s}]$$

from drift time data, plot the apparent pad response function as a function of the drift distance. Then perform a straight-line fit.

From the slope

$$C_D = 128\,[\mu\mathrm{m}/\sqrt{\mathrm{cm}}]$$

From the intercept

$$\sigma^2_{\mathrm{PR}}(0) := \sigma^2_{\mathrm{PR}} - C_d^2 z = \sigma^2_{\mathrm{PRF}} + \frac{w^2}{12}$$

$$\begin{aligned} \sigma^2_{PRF} &= \sigma_{PR}(0)^2 - \frac{w^2}{12} \\ &= (270\,[\mu\mathrm{m}])^2 \end{aligned}$$

roughly consistent with what we expect from the diffusion in transfer and induction gaps.

Angular Pad Effect

Resolution degradation for inclined tracks

Consider an inclined track having an angle phi to the yz plane and an angle theta to the xy plane (pad plane). The projection of the track electrons to the xz plane is no longer point-like even if the cluster size is negligible for secondary ionizations. This extra charge spread adds up to that caused by diffusion. Consequently, the statistical fluctuation of the locations of the primary ionizations as well as that of the 2ndary ionizations cause additional contributions to the coordinate measurement error. The effect is further amplified by the gas gain fluctuations. The degradation of spatial resolution due to the finite phi is known as the angular pad effect and is inevitable as long as we use ordinary readout pads, since they break the rotational symmetry in the phi direction (notice that the symmetry breaking must be much softer in the case of pixel readout). If the theta is nonzero, the drift distance depends on where you are on the track and the average number of ionization electrons will be larger due to the longer track segment per pad row. As long as we use a short enough pad, the drift distance can be regarded as approximately constant within a pad row. We can hence assume that the effect of the finite theta can be taken into account by scaling Neff by the amount expected from the increase of the track segment length. For this reason we assume in what follows that the theta is zero unless otherwise stated. We again start from an ideal situation.

Ionization Statistics

Ideal Readout Plane: Coordinate = Simple C.O.G.

PDF for C.O.G.

We assume here an ideal readout plane that can measure the x-coordinates of individual track electrons exactly. The probability distribution function for the center of gravity is given by

Primary ionization statistics

Cluster size distribution

$$P(\bar{x}) = \sum_{N=1}^{\infty} P_{PI}(N) \prod_{i=1}^{N} \left[\int \frac{ds_i}{l} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right.$$
$$\left. \times \prod_{j=1}^{M_i} \int d\Delta x_{ij} \, d\Delta y_{ij} \, P_D(\Delta x_{ij}) \, P_D(\Delta y_{ij}) \right]$$
$$\times \, \delta \left(\bar{x} - \frac{\sum_{i,j} x_{ij} \, \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)}{\sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)} \right)$$

Gaussian diffusion

Ideal readout plane

$$P_D(x_i; \sigma_d) = \frac{1}{\sqrt{2\pi}\sigma_d} \exp\left(- \frac{x_i^2}{2\sigma_d^2} \right)$$

$$\sigma_d = C_d \sqrt{z}$$

where Cd is the diffusion coefficient and z is the drift length. The "l" is the projected track length to the xy plane and s_i is the projected location of i-th cluster along the track. The arrival point of j-th electron in the i-th cluster is (x_ij, y_ij):

$$x_{ij} = \tilde{x} + s_i \, \sin\phi + \Delta x_{ij}$$
$$y_{ij} = s_i \, \cos\phi + \Delta y_{ij}$$

where s_i=0 is in the middle of the row in question. The "L" is the height of the row. The average x-coordinate of the center of gravity of the track electrons arriving at the row is

$$\langle \bar{x} \rangle := \int d\bar{x} \, P(\bar{x}) \, \bar{x} = \tilde{x}$$

since s_i, delta x_ij, and delta Y_ij average to zero.

Variance of the C.O.G.

By definition the variance of the C.O.G. is given by

$$\sigma_{\bar{x}}^2 := \int d\bar{x}\, P(\bar{x})\, (\bar{x} - \langle \bar{x} \rangle)^2$$

Substituting

$$\langle \bar{x} \rangle = \tilde{x}$$
$$x_{ij} = \tilde{x} + s_i\, \sin\phi + \Delta x_{ij}$$

we have

$$\sigma_{\bar{x}}^2 = \sum_{N=1}^{\infty} P_{PI}(N) \prod_{i=1}^{N} \left[\int \frac{ds_i}{l} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right. \\ \times \left. \prod_{j=1}^{M_i} \int d\Delta x_{ij}\, d\Delta y_{ij}\, P_D(\Delta x_{ij})\, P_D(\Delta y_{ij}) \right] \\ \times \left(\frac{\sum_{i,j} (s_i\, \sin\phi + \Delta x_{ij})\, \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)}{\sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)} \right)^2$$

The delta x_ij integral is straightforward

$$\sigma_{\bar{x}}^2 = \sum_{N=1}^{\infty} P_{PI}(N) \prod_{i=1}^{N} \left[\int \frac{ds_i}{l} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \prod_{j}^{M_i} \int d\Delta y_{ij}\, P_D(\Delta y_{ij}) \right] \\ \times \left[\frac{\sin^2\phi \sum_i s_i^2 \left(\sum_j \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right) \right)^2 + \sigma_d^2 \sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)}{\left(\sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right) \right)^2} \right]$$

since cross terms just vanish.

We can cast this into the form:

$$\sigma_{\bar{x}}^2 = \sin^2\phi \left\langle \frac{\sum_i s_i^2 \left(\sum_j \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right) \right)^2}{\left(\sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right) \right)^2} \right\rangle \\ + \sigma_d^2 \left\langle \frac{1}{\sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)} \right\rangle$$

where the 1st term on the R.H.S. is given by

$$\sigma_{\bar{x};\mathrm{ang}}^2 = \sin^2\phi \sum_{N=1}^{\infty} P_{PI}(N) \prod_{i=1}^{N} \left[\int \frac{ds_i}{l} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right. \\ \times \left. \prod_{j=1}^{M_i} \int d\Delta y_{ij}\, P_D(\Delta y_{ij}) \right] \frac{\sum_i s_i^2 \left(\sum_j \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right) \right)^2}{\left(\sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right) \right)^2}$$

while the 2nd term by

$$\sigma_{\bar{x};\mathrm{diff}}^2 = \sigma_d^2 \sum_{N=1}^{\infty} P_{PI}(N) \prod_{i=1}^{N} \left[\int \frac{ds_i}{l} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right. \\ \times \left. \prod_{j=1}^{M_i} \int d\Delta y_{ij}\, P_D(\Delta y_{ij}) \right] \frac{1}{\sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)}$$

Notice that the 2nd term defines Neff by

$$\frac{1}{N_{\mathrm{eff}}} := \sum_{N=1}^{\infty} P_{PI}(N) \prod_{i=1}^{N} \left[\int \frac{ds_i}{l} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right. \\ \times \left. \prod_{j=1}^{M_i} \int d\Delta y_{ij}\, P_D(\Delta y_{ij}) \right] \frac{1}{\sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)}$$

which is a more microscopic definition in terms of the combined effects of primary and 2ndary ionization statistics together with diffusions. It's worth noting that

$$N_{\rm acc} := \sum_{i,j} \theta\left(\frac{L}{2} + y_{ij}\right) \theta\left(\frac{L}{2} - y_{ij}\right)$$

just counts the number of track electrons accepted by the row in question. If the track is far away from the readout plane, the memory about parent clusters will be lost by the time individual track electrons arrive at the readout plane because of diffusions. On the other hand, if the track is near the readout plane, the correlation among the track electrons belonging to the same parent cluster is very strong and hence they are either all accepted or all rejected by the row and hence the Neff value will be smaller than those at longer drift distances. This clustering effect should be there even in the case of phi=0. Neff is in principle z-dependent!

The probability of a 2ndary electron at s_i reaches the row in question is given by

$$\eta(s_i \, \cos\phi) := \int_{-\frac{L}{2} - s_i \, \cos\phi}^{+\frac{L}{2} - s_i \, \cos\phi} \frac{d\Delta y}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{(\Delta y)^2}{2\sigma_d^2}\right)$$

With this the probability of k electrons created at s_i reach the row is written in the form:

$$\bar{P}_{SI}(k; y_i) := \sum_{M=1}^{\infty} P_{SI}(M) \frac{M!}{k!(M-k)!} \eta(y_i)^k (1 - \eta(y_i))^{M-k}$$

Then we can rewrite the P.D.F. for the C.O.G. as

$$P(\bar{x}) = \sum_{N=1}^{\infty} P_{PI}(N) \prod_{i=1}^{N} \left[\int \frac{ds_i}{l} \sum_{k_i=0}^{\infty} \bar{P}_{SI}(k_i; s_i \cos\phi) \prod_{j=1}^{k_i} \int d\Delta x_{ij} \, P_D(\Delta x_{ij}; \sigma_d) \right] \times \delta\left(\bar{x} - \frac{\sum_{i=1}^{N} \sum_{j=1}^{k_i} (\tilde{x} + s_i \, \sin\phi + \Delta x_{ij})}{\sum_{i=1}^{N} k_i}\right)$$

where

$$P_{PI}(N) := \frac{\bar{N}^N}{N!} \exp(-\bar{N})$$

and

$$P_D(\Delta x; \sigma_d) := \frac{1}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{(\Delta x)^2}{2\sigma_d^2}\right)$$

From these we have

$$\sigma^2_{\bar{x}:\mathrm{diff}} = \sigma_d^2 \sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1,\cdots,k_N} \prod_{i=1}^{N} \left[\int \frac{dy_i}{l\cos\phi} \bar{P}_{SI}(k_i;y_i) \right] \left(\frac{1}{\sum_{i=1}^{N} k_i} \right)$$

which implies

$$N_{\mathrm{eff}} := \left[\sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1,\cdots,k_N} \prod_{i=1}^{N} \left(\int \frac{dy_i}{l\cos\phi} \bar{P}_{SI}(k_i;y_i) \right) \left(\frac{1}{\sum_{i=1}^{N} k_i} \right) \right]^{-1}$$

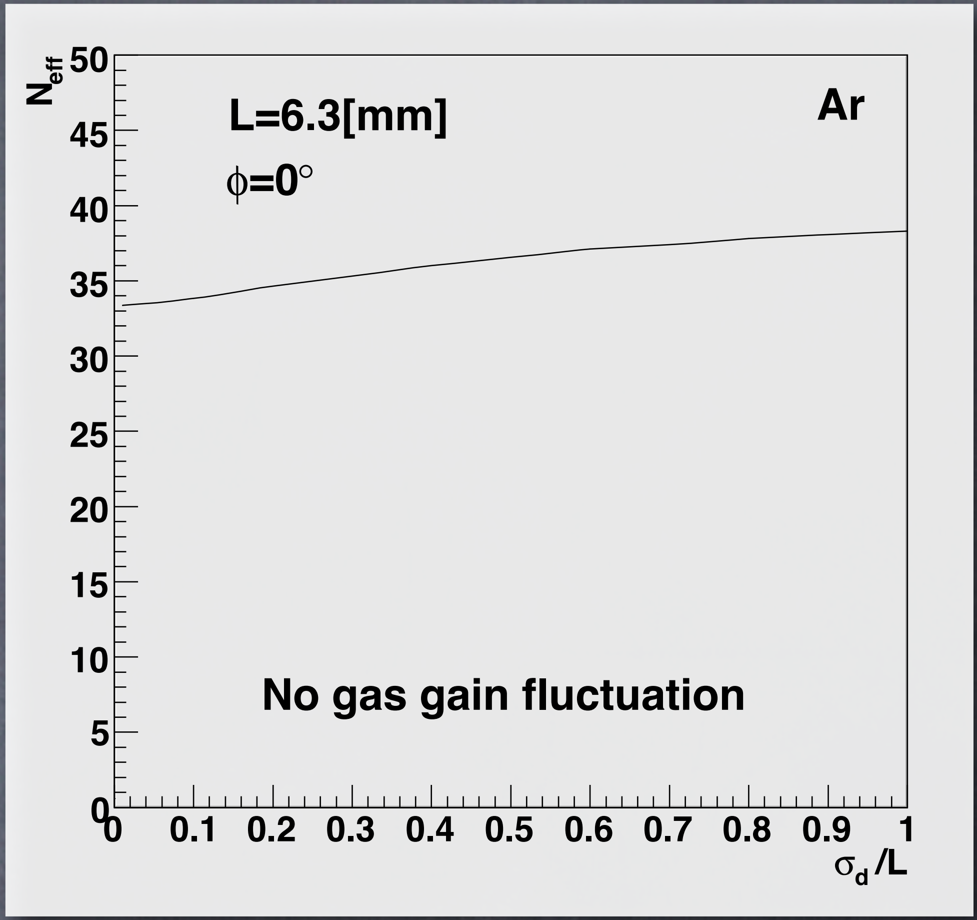
Defining

$$\bar{\bar{P}}(k_i) := \int \frac{dy}{l\cos\phi} \bar{P}_{SI}(k_i;y)$$

We can further reduce this formula for Neff to

$$N_{\mathrm{eff}} = \left[\sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1,\cdots,k_N} \prod_{i=1}^{N} \left(\bar{\bar{P}}_{SI}(k_i) \right) \left(\frac{1}{\sum_{i=1}^{N} k_i} \right) \right]^{-1}$$

For a row with a height of L=6.3mm, Neff has been calculated as a function of the relative diffusion, sigma_d/L, assuming Ar 100% and normal incidence and plotted in the following figure. We can see the effect of de-clustering that appears as a slight increase of Neff with the drift distance, being consistent with our naive expectation.



Notice that for an LC-TPC, for which the Cd value is expected to be less than 50 microns/sqrt(cm), sigma_d/L never exceeds 0.2 even for a drift length greater than 200 cm.

The Neff value is hence approximately the drift-length independent, justifying our analytic formula for the phi=0 case.

Angular effect

The angular effect is contained in

$$\sigma^2_{\bar{x}:\mathrm{ang}} = \tan^2\phi \sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1,\cdots,k_N} \prod_{i=1}^{N} \left[\int \frac{dy_i}{l\cos\phi} \bar{P}_{SI}(k_i;y_i)\right] \frac{\sum_{i=1}^{N} k_i^2 y_i^2}{\left(\sum_{i=1}^{N} k_i\right)^2}$$

Defining

$$\langle y^2 \rangle_{k_i} := \int \frac{dy}{l\cos\phi} \bar{P}_{SI}(k_i;y)\, y^2$$

and recalling

$$\bar{\bar{P}}(k_i) := \int \frac{dy}{l\cos\phi} \bar{P}_{SI}(k_i;y)$$

we can rewrite this formula as

$$\begin{aligned} \sigma^2_{\bar{x}:\mathrm{ang}} &= \tan^2\phi \sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1,\cdots,k_N} \sum_{i'=1}^{N} \prod_{i\neq i'} \left[\bar{\bar{P}}(k_i)\right] \frac{k_{i'}^2 \langle y^2 \rangle_{k_{i'}}}{\left(\sum_{i=1}^{N} k_i\right)^2} \\ &= \tan^2\phi \sum_{N=1}^{\infty} P_{PI}(N)\, N \sum_{k_1,\cdots,k_N} \prod_{i=2}^{N} \left[\bar{\bar{P}}(k_i)\right] \frac{k_1^2 \langle y^2 \rangle_{k_1}}{\left(\sum_{i=1}^{N} k_i\right)^2} \end{aligned}$$

Notice that

$$P_{PI}(N) := \frac{\bar{N}^N}{N!} \exp(-\bar{N})$$

depends on ϕ through $\bar{N}$. On the other hand, $\langle y^2 \rangle_k$ and $\bar{\bar{P}}(k)$ are angle-independent since

$$l\cos\phi = y_{\max} - y_{\min} =: \Delta Y$$

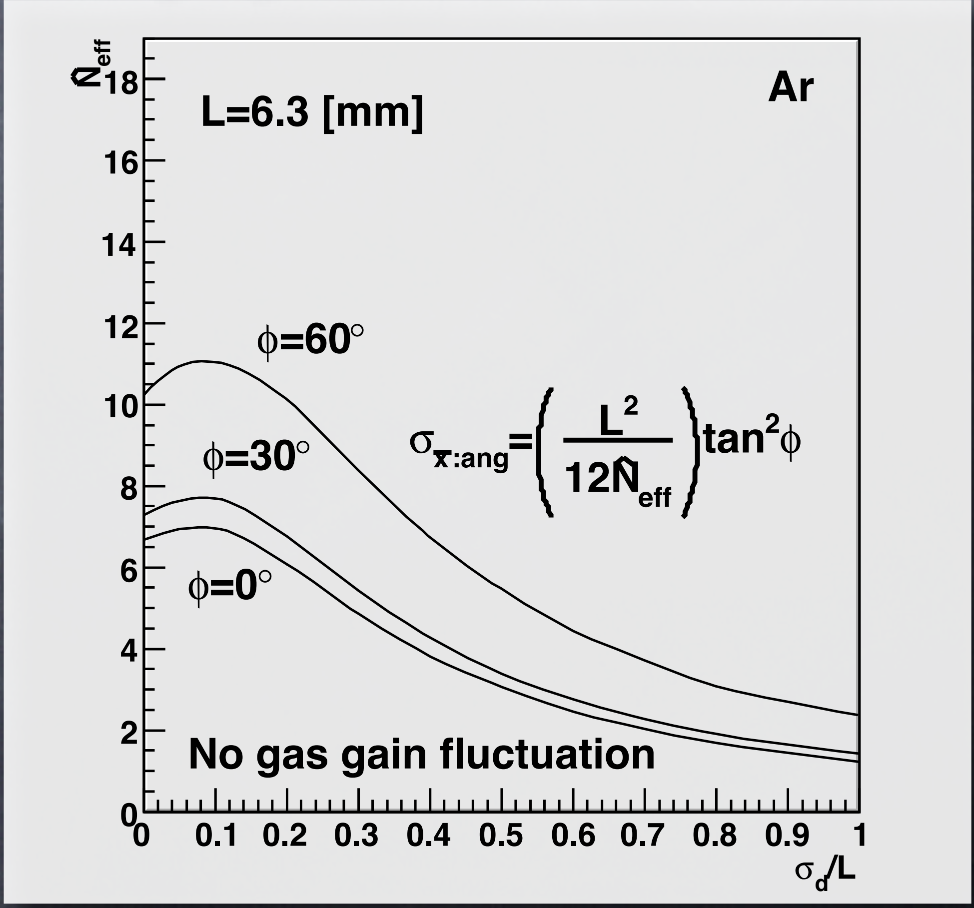
It is probably useful to further rewrite the formula in the following form:

$$\sigma^2_{\bar{x}:\mathrm{ang}} = \frac{L^2}{12\,\hat{N}_{\mathrm{eff}}} \tan^2\phi$$

where

$$\hat{N}_{\mathrm{eff}} := \left[\frac{12}{L^2} \sum_{N=1}^{\infty} P_{PI}(N)\, N \sum_{k_1,\cdots,k_N} \prod_{i=2}^{N} \left[\bar{\bar{P}}(k_i)\right] \frac{k_1^2 \langle y^2 \rangle_{k_1}}{\left(\sum_{i=1}^{N} k_i\right)^2}\right]^{-1}$$

The following is a sample calculation for L=6.3 [mm].



[I] Short drift limit

In the short drift limit, the diffusion becomes negligible and its corresponding P.D.F. becomes a delta-function and hence we have

$$\eta(y) \to \theta\left(\frac{L}{2}+y\right)\theta\left(\frac{L}{2}-y\right) \quad \text{as} \quad \frac{\sigma_d}{L} \to 0$$

leading us to

$$\bar{P}_{SI}(k;y_i) := \sum_{M=1}^{\infty} P_{SI}(M)\,\frac{M!}{k!(M-k)!}\eta(y_i)^k(1-\eta(y_i))^{M-k}$$

$$\to P_{SI}(k)\,\theta\left(\frac{L}{2}+y\right)\theta\left(\frac{L}{2}-y\right)+\delta_{k,0}\,\theta\left(-\frac{L}{2}+y\right)\theta\left(y-\frac{L}{2}\right)$$

From this we have

$$\bar{\bar{P}}(k) := \int \frac{dy}{\Delta Y}\,\bar{P}_{SI}(k;y) \to \frac{L}{\Delta Y}\,P_{SI}(k)+\left(1-\frac{L}{\Delta Y}\right)\delta_{k,0}$$

and

$$\left\langle y^2\right\rangle_k := \int \frac{dy}{\Delta Y}\,\bar{P}_{SI}(k;y)\,y^2$$

$$\to \left(\frac{L}{\Delta Y}\right)P_{SI}(k)\frac{L^2}{12}+\left(\frac{(\Delta Y)^2}{12}-\frac{L}{\Delta Y}\frac{L^2}{12}\right)\delta_{k,0}$$

Notice that L/Deta Y is the probability of a primary cluster is created within the row in question. We hence define

$$\bar{\eta} := \frac{L}{\Delta Y}$$

With this we have

$$\begin{aligned}
\left[\hat{N}_{\rm eff}\right]^{-1} &:= \frac{12}{L^2}\sum_{N=1}^{\infty} P_{PI}(N)\,N \sum_{k_1,\cdots,k_N}\prod_{i=2}^{N}\left[\bar{\bar{P}}(k_i)\right]\frac{k_1^2\,\left\langle y^2\right\rangle_{k_1}}{\left(\sum_{i=1}^N k_i\right)^2} \\
&\to \sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1,\cdots,k_N}\prod_{i=2}^{N}\left[P_{SI}(k_i)\,\bar{\eta}+(1-\bar{\eta})\,\delta_{k_i,0}\right]\frac{N k_1^2\bar{\eta}\,P_{SI}(k_1)}{\left(\sum_{i=1}^N k_i\right)^2} \\
&= \sum_{N=1}^{\infty} P_{PI}(N)\sum_{N'=1}^{N}\bar{\eta}^{N'}(1-\bar{\eta})^{N-N'}\,{}_N C_{N'} \sum_{k_1,\cdots,k_{N'}>0}\prod_{i=1}^{N'}\left[P_{SI}(k_i)\right]\frac{\sum_{i=1}^{N'} k_i^2}{\left(\sum_{i=1}^{N'} k_i\right)^2}
\end{aligned}$$

where in the last line N' is the number of clusters created within the y range of the row in question. We can hence obtain

$$\hat{N}_{\rm eff} \simeq \left[\left\langle \frac{\sum_{i=1}^{N'} M_i^2}{\left(\sum_{i=1}^{N'} M_i\right)^2}\right\rangle\right]^{-1}$$

in the short drift limit. Notice that the effective number of track electrons for the angle term is determined by primary ionization statistics for N' and secondary ionization statistics for M_i. If L and hence <N'> is large enough for the approximation:

$$\sum_{i=1}^{N'} M_i \approx N'\left\langle M\right\rangle =: N'\bar{M}$$

then we arrive at

$$\lim_{z\to 0} N_{\rm eff} = \left[\left\langle\frac{1}{N_{PI}}\right\rangle\left\langle\left(\frac{M}{\bar{M}}\right)^2\right\rangle\right]^{-1}$$

This last formula has exactly the same form as the effect of gas gain fluctuation. Since the secondary ionization has a long tail and the number of primary electrons per row is less than 20 for L=6.3 mm, the approximation is expected to be bad. Nevertheless, the formula suggests that the effective number of electrons for the angle term is much smaller than that for the diffusion term as seen from the sample calculation.

[II] Long drift limit

In the long drift limit (sigma_d/L>>1), the diffusion dominates the row height and we have

$$
\begin{aligned}
\eta(y) \;&:=\; \int_{-\frac{L}{2}}^{+\frac{L}{2}} \frac{dy'}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{(y'-y)^2}{2\sigma_d^2}\right) \\
&\to\; \frac{L}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{y^2}{2\sigma_d^2}\right) \quad \text{as} \quad \frac{\sigma_d}{L} \to \infty
\end{aligned}
$$

Notice that this probability is infinitesimal in the limit and hence we can safely ignore the probability of more than one electron from the same cluster reaches the row in question is negligible. We hence obtain

$$
\begin{aligned}
\bar{P}_{SI}(k;y) \;&:=\; \sum_{M=1}^{\infty} P_{SI}(M) \frac{M!}{k!(M-k)!} \eta(y)^k (1-\eta(y))^{M-k} \\
&\to\; \sum_{M=k}^{\infty} P_{SI}(M) \left[(1-\eta)^M \delta_{k,0} + M\eta\,(1-\eta)^{M-1} \delta_{k,1}\right] \\
&\simeq\; \sum_{M=k}^{\infty} P_{SI}(M) \left[(1-M\eta)\delta_{k,0} + M\eta\delta_{k,1}\right]
\end{aligned}
$$

leading us to

$$
\bar{P}_{SI}(k;y) \simeq (1-\bar{M}\eta(y))\delta_{k,0} + \bar{M}\eta(y)\delta_{k,1}
$$

Using this we have

$$
\bar{\bar{P}}(k) := \int \frac{dy}{\Delta Y}\, \bar{P}_{SI}(k;y) \to \left(1-\bar{M}\bar{\eta}\right)\delta_{k,0} + \bar{M}\bar{\eta}\,\delta_{k,1}
$$

and

$$
\begin{aligned}
\langle y^2 \rangle_k \;&:=\; \int \frac{dy}{\Delta Y}\, \bar{P}_{SI}(k;y)\, y^2 \\
&\to\; \left(\frac{(\Delta Y)^2}{12} - \bar{M}\bar{\eta}\sigma_d^2\right)\delta_{k,0} + \bar{M}\bar{\eta}\,\sigma_d^2\,\delta_{k,1}
\end{aligned}
$$

with

$$
\bar{\eta} := \frac{L}{\Delta Y}
$$

where we have assumed that Delta Y is big enough compared to sigma_d. Using these formulae we can now calculate Neff for the angle term.

$$
\begin{aligned}
\left[\hat{N}_{\rm eff}\right]^{-1} &:= \frac{12}{L^2}\sum_{N=1}^{\infty} P_{PI}(N)\, N \sum_{k_1,\cdots,k_N}\prod_{i=2}^{N}\left[\bar{\bar{P}}(k_i)\right] \frac{k_1^2\,\left\langle y^2\right\rangle_{k_1}}{\left(\sum_{i=1}^{N} k_i\right)^2} \\
&\to \frac{12\,\sigma_d^2}{L^2}\sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1,\cdots,k_N}\prod_{i=2}^{N}\left[(1-\bar{M}\bar{\eta})\,\delta_{k_i,0}+\bar{M}\bar{\eta}\,\delta_{k_i,1}\right] \frac{N\,k_1^2\,\bar{M}\bar{\eta}}{\left(\sum_{i=1}^{N} k_i\right)^2} \\
&= \frac{12\,\sigma_d^2}{L^2}\sum_{N=1}^{\infty} P_{PI}(N) \sum_{K=1}^{N}\sum_{k_1,\cdots,k_K}\prod_{i=1}^{K}\left[(\bar{M}\bar{\eta})^K(1-(\bar{M}\bar{\eta}))^{N-K}\,{}_N C_K\right]\frac{1}{K}
\end{aligned}
$$

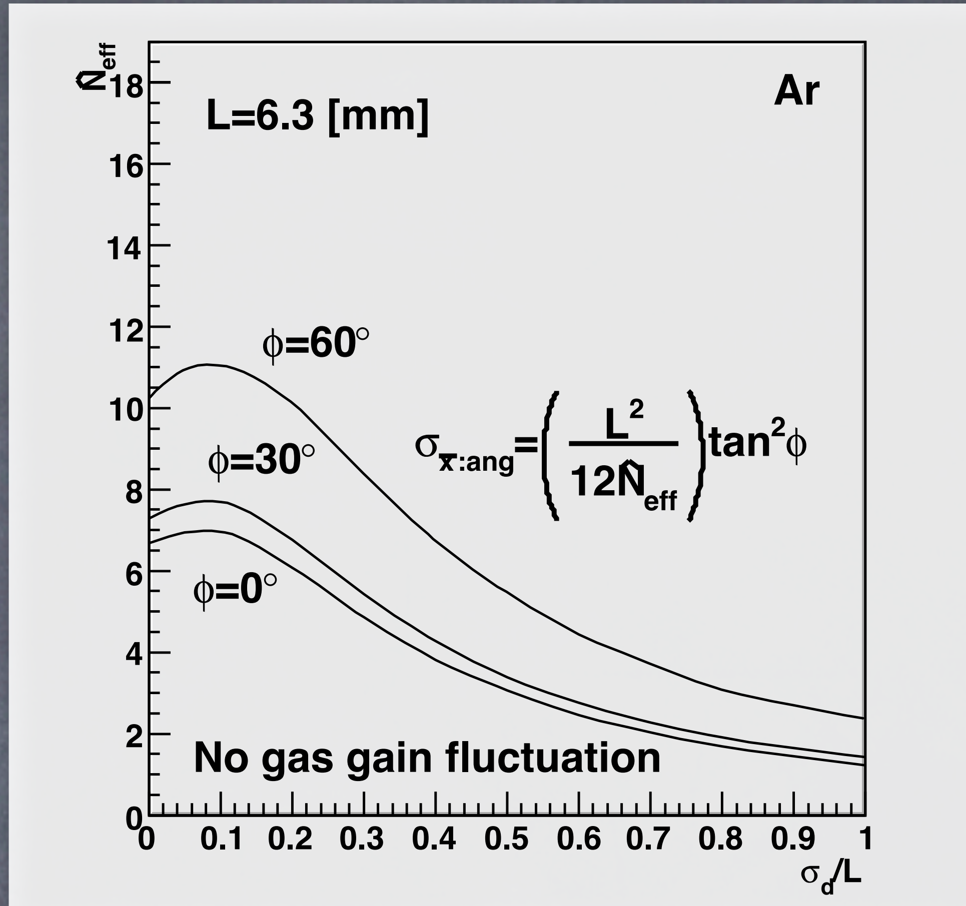
where K is the number of electrons arrived at the row in question. We can hence write

$$
[\hat{N}_{\rm eff}]^{-1} = \lim_{\frac{\sigma_d}{L}\to\infty} \frac{12\,\sigma_d^2}{L^2}\left\langle \frac{1}{K}\right\rangle
$$

The effective number of track electrons for the angle term hence decreases with the drift length because of the increase of sigma_d as seen in the sample calculation. As a matter of fact, we can rewrite the angle term in the large drift limit as

$$
\sigma^2_{\bar{x}:{\rm ang}} \to \tan^2\phi\left\langle\frac{1}{K}\right\rangle\sigma_d^2
$$

which indicates that diffusion dominates also in the y-direction. For convenience I show the same sample calculation here again. We see slight increase of the Neff



at small drift distance due to de-clustering followed by monotonic decrease at longer drift distance. If you make the row too narrow, you cannot benefit because of the diffusion in the y direction. The figure suggests that we have to keep the diffusion for the maximum drift below 0.15. L=6.3mm for the LC-TPC seems reasonable.

Gas Gain Fluctuation

Coordinate = Gain-Weighted Mean

PDF for gain-weighted mean

We now switch on the gas gain fluctuation and assume that the coordinate measured by the readout plane is the gain-weighted mean of the N ionization electrons.

Primary ionization statistics

Cluster size distribution

$$P(\bar{x}) = \sum_{N=1}^{\infty} P_{PI}(N) \prod_{i=1}^{N} \left[\int \frac{dy_i}{\Delta Y} \sum_{k_i=0}^{\infty} \bar{P}_{SI}(k_i; y_i) \right.$$
$$\left. \prod_{j=1}^{k_i} \left(\int d\Delta x_{ij}\, P_D(\Delta x_{ij}; \sigma_d) \int dG_{ij}\, P_G\left(\frac{G_{ij}}{\bar{G}}\right)\right)\right]$$
$$\times\ \delta\left(\bar{x} - \frac{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}(\tilde{x} + y_i \tan\phi + \Delta x_{ij})}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)$$

Gaussian diffusion

Gain-weighted mean

$$P_D(x_i; \sigma_d) = \frac{1}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{x_i^2}{2\sigma_d^2} \right)$$

$$\sigma_d = C_d \sqrt{z}$$

where Cd is the diffusion coefficient and z is the drift length. G_ij is the gas gain for j-the electron in i-th cluster whose P.D.F. is given by P_G as before.

Recall that the primary ionization statistics is governed by

$$P_{PI}(N) := \frac{\bar{N}^N}{N!} \exp(-\bar{N})$$

and effective cluster size distribution by

$$\bar{P}_{SI}(k; y_i) := \sum_{M=1}^{\infty} P_{SI}(M) \frac{M!}{k!(M-k)!} \eta(y_i)^k (1 - \eta(y_i))^{M-k}$$

where k is the number of electrons that is accepted by the row in question starting from the i-th cluster created at y=y_i.

Since Y_i and delta x_ij have no bias, we obviously have

$$\langle \bar{x} \rangle := \int d\bar{x}\, P(\bar{x})\, \bar{x} = \tilde{x}$$

Variance of the G.W.M.

By definition the variance of the gain-weighted mean is given by

$$\sigma_{\bar{x}}^2 := \int d\bar{x}\, P(\bar{x})\, (\bar{x} - \langle \bar{x} \rangle)^2$$

Substituting

$$\langle \bar{x} \rangle = \tilde{x}$$

we have

$$\begin{aligned}
\sigma_{\bar{x}}^2 &= \left\langle \left(\frac{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}\, (y_i \tan\phi + \Delta x_{ij})}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle \\
&= \sigma_d^2 \left\langle \frac{\sum_{i,j} G_{ij}^2}{\left(\sum_{i,j} G_{ij}\right)^2} \right\rangle + \tan^2\phi \left\langle \left(\frac{\sum_{i=1}^N y_i \sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^N \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle
\end{aligned}$$

Making the same approximation

$$\sum_{i,j} G_{ij} \approx \bar{G} \sum_{i=1}^N k_i$$

as we did in the phi=0 case, we obtain

$$\sigma_{\bar{x}:\mathrm{diff}}^2 := \sigma_d^2 \left\langle \frac{\sum_{i,j} G_{i,j}^2}{\left(\sum_{i,j} G_{i,j}\right)^2} \right\rangle \simeq \sigma_d^2 \left\langle \frac{1}{\sum_i k_i} \right\rangle \left\langle \left(\frac{G}{\bar{G}} \right)^2 \right\rangle$$

for the angle-independent term which is none other than the formula we derived before for the phi=0 case.

For the angle-dependent term we have

$$\begin{aligned}
\sigma_{\bar{x}:\mathrm{ang}}^2 &:= \tan^2\phi \left\langle \left(\frac{\sum_{i=1}^N y_i \sum_{j=1}^{k_i} G_{i,j}}{\sum_{i=}^N \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle \\
&\simeq \tan^2\phi \left[\left\langle \frac{\sum_i k_i^2\, y_i^2}{(\sum_i k_i)^2} \right\rangle + \left\langle \frac{\sum_i k_i\, y_i^2}{(\sum_i k_i)^2} \right\rangle \sigma_{(\frac{G}{\bar{G}})}^2 \right]
\end{aligned}$$

[I] Short drift limit

In the short drift limit, the angle term becomes

$$\sigma_{\bar{x}:\mathrm{ang}}^2 \simeq \tan^2\phi \left[\left\langle \frac{\sum_i M_i^2}{(\sum_i M_i)^2} \right\rangle + \left\langle \frac{1}{\sum_i M_i} \right\rangle \sigma_{(\frac{G}{\bar{G}})}^2 \right] \frac{L^2}{12}$$

since there is no diffusion in this limit and hence we can replace k_i by M_i and the average of y_i^2 becomes independent of k_i and just gives L^2/12. The formula implies the effective number for the angle term being

$$\hat{N}_{\mathrm{eff}} \simeq \left[\left\langle \frac{\sum_i M_i^2}{(\sum_i M_i)^2} \right\rangle + \left\langle \frac{1}{\sum_i M_i} \right\rangle \sigma_{(\frac{G}{\bar{G}})}^2 \right]^{-1}$$

where the 2nd term in the square bracket is from gain fluctuation.

[II] Long drift limit

We restart from

$$\sigma^2_{\bar{x}:\mathrm{ang}} := \tan^2\phi \left\langle \left(\frac{\sum_{i=1}^{N} y_i \sum_{j=1}^{k_i} G_{i,j}}{\sum_{i=}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle$$

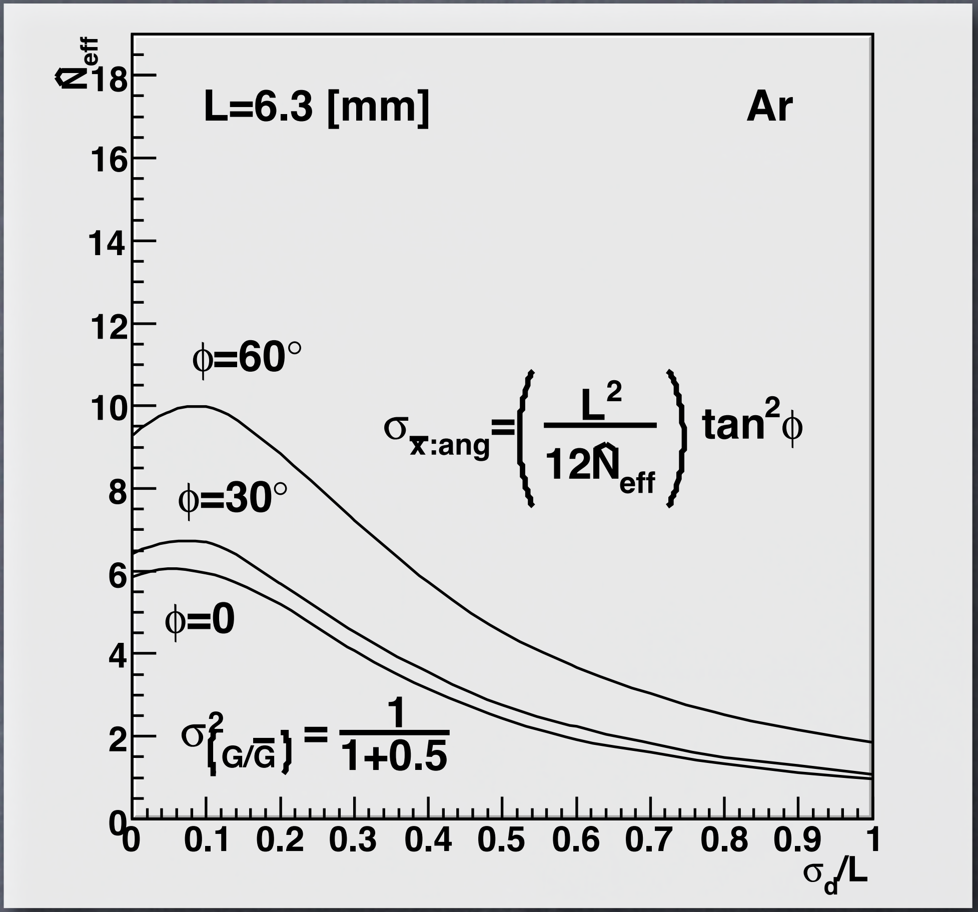
$$\simeq \tan^2\phi \left[\left\langle \frac{\sum_i k_i^2\, y_i^2}{(\sum_i k_i)^2} \right\rangle + \left\langle \frac{\sum_i k_i\, y_i^2}{(\sum_i k_i)^2} \right\rangle \sigma^2_{\left(\frac{G}{\bar{G}}\right)} \right]$$

Recalling that in the long drift limit the row height becomes negligible and the effective cluster size k_i for the i-th cluster can be at most 1 and that each electron accepted by the row must have experienced average diffusion of sigma_d^2, we have

$$\sigma^2_{\bar{x}:\mathrm{ang}} \simeq \tan^2\phi \left[\left\langle \frac{1}{\sum_i k_i} \right\rangle + \left\langle \frac{1}{\sum_i k_i} \right\rangle \sigma^2_{\left(\frac{G}{\bar{G}}\right)} \right] \sigma_d^2$$

$$\simeq \tan^2\phi \left\langle \frac{1}{K} \right\rangle \left\langle \left(\frac{G}{\bar{G}} \right)^2 \right\rangle \sigma_d^2$$

where K is the number of electrons arrived at the row in question. This matches our naive expectation from the phi=0 formula for the x-resolution.



In the short drift region, the gas gain fluctuation only slightly reduces the Neff for the angle term (10% effect). Since the sigma_d/L will not exceed 0.2 for the LC-TPC, we can conclude that the angle term is rather insensitive to the gain fluctuation.

Finite Size Pads

Coordinate = Charge Centroid

PDF for charge centroid

We now replace the continuous readout plane with an array of finite size pads. The finite size pads break the translational symmetry. We hence need to specify the track position relative to the pad center. The arrival point of j-th ionization electron from i-th primary cluster is given by

$$\underbrace{x_{ij}}_{} = \underbrace{\tilde{x}}_{\text{track position}} + \overbrace{y_i}^{\text{i-th cluster y position}} \tan\phi + \Delta x_{ij}$$

$$y_{ij} = y_i + \Delta y_{ij}$$

diffusion : $\langle(\Delta x_{ij})^2\rangle = \langle(\Delta y_{ij})^2\rangle = \sigma_d^2 = C_d^2\, z$

The charge in units of electron charge on a-th pad is given by

$$Q_a = \sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij}\, F_a(x_{ij}, y_{ij}) + \Delta Q_a$$

where F_a is the normalized pad response function for a-th pad as before and G_{ij} is the gas gain for the j-th ionization electron from the i-th primary cluster, and ΔQ_a is the electronic noise:

$$\langle(\Delta Q_a)^2\rangle = \sigma_E^2$$

The charge centroid is then given by

$$\bar{x} = \sum_a Q_a\,(a\,w) \Big/ \sum_a Q_a$$

with w being the pad pitch. The probability distribution function for charge centroid is

$$P(\bar{x};\tilde{x}) = \sum_{N=1}^{\infty} P_{PI}(N) \sum_{M_1,\cdots,M_N} \prod_{i=1}^{N} \Bigg[\int \frac{dy_i}{\Delta Y} P_{SI}(M_i) \prod_{j=1}^{M_i} \left(\int d\Delta x_{ij}\, P_D(\Delta x_{ij};\sigma_d) \int d\Delta y_{ij}\, P_D(\Delta y_{ij};\sigma_d) \int dG_{ij}\, P_G\left(\frac{G_{ij}}{\bar{G}}\right)\right)\Bigg] \prod_a \Bigg[\int d\Delta Q_a\, P_E(\Delta Q_a;\sigma_E) \int dQ_a\, \delta\left(Q_a - \sum_{i=1}^{N}\sum_{j=1}^{M_i} G_{ij}\, F_a(x_{ij}, y_{ij}) - \Delta Q_a\right)\Bigg] \times \delta\left(\bar{x} - \frac{\sum_a Q_a\,(aw)}{\sum_a Q_a}\right)$$

Variance of charge centroid

In order to take into account the effect of finite size pads as known as the S-shape systematics, we define the variance by

$$\sigma_{\bar{x}}^2 := \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\left(\frac{\tilde{x}}{w}\right) \int d\bar{x}\, P(\bar{x};\tilde{x})\,(\bar{x}-\tilde{x})^2$$

as with the phi=0 case. Substituting the PDF given above in this and with some arithmetics, we obtain

$$\sigma_{\bar{x}}^2 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\left(\frac{\tilde{x}}{w}\right) \left[[A'] + [B'] \left\langle \left(\frac{G}{\bar{G}}\right)^2 \right\rangle \right] + [C]$$

with [A'], [B'], and [C] corresponding to [A], [B], and [C] for the phi=0 case.
[A'] is independent of gas gain and given by

$$\begin{aligned}
[A'] := \sum_{a,b} (ab\,w^2) \Bigg[&\left\langle \frac{\sum_i k_i^2 \left(\langle\langle F_a\rangle \langle F_b\rangle\rangle_{k_i} - \langle\langle F_a\rangle\rangle_{k_i} \langle\langle F_b\rangle\rangle_{k_i} \right)}{\left(\sum_i k_i\right)^2} \right\rangle \\
&+ \left\langle \left(\frac{\sum_i k_i \langle\langle F_a\rangle\rangle_{k_i}}{\sum_i k_i} \right) \left(\frac{\sum_i k_i \langle\langle F_b\rangle\rangle_{k_i}}{\sum_i k_i} \right) \right\rangle \\
&- \left\langle \frac{\sum_i k_i \langle\langle F_a\rangle\rangle_{k_i}}{\sum_i k_i} \right\rangle \left\langle \frac{\sum_i k_i \langle\langle F_b\rangle\rangle_{k_i}}{\sum_i k_i} \right\rangle \Bigg] \\
&+ \left[\sum_a (a\,w) \left\langle \frac{\sum_i k_i \langle\langle F_a\rangle\rangle_{k_i}}{\sum_i k_i} \right\rangle - \tilde{x} \right]^2
\end{aligned}$$

where we have defined

$$\begin{aligned}
\langle F_a\rangle &:= \int d\Delta x\, P_D(\Delta x;\sigma_d) \\
&\quad \times F_a(\tilde{x} + y\tan\phi + \Delta x)
\end{aligned}$$

$$\langle\langle F_a\rangle\rangle_k := \frac{1}{\bar{\bar{P}}(k)} \int \frac{dy}{\Delta Y}\, \bar{P}_{SI}(k;y)\,\langle F_a\rangle$$

$$\langle\langle F_a\rangle \langle F_b\rangle\rangle_k := \frac{1}{\bar{\bar{P}}(k)} \int \frac{dy}{\Delta Y}\, \bar{P}_{SI}(k;y)\,\langle F_a\rangle \langle F_b\rangle$$

and the outermost average is taken as

$$\langle[\cdots]\rangle := \sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1,\cdots,k_N} \prod_{i=1}^{N} \left(\bar{\bar{P}}(k_i)\right) [\cdots]$$

In the de-clustering limit, only k_i=1 counts and [A'] becomes

$$\begin{aligned}
\lim_{\frac{\sigma_d}{L}\to\infty} [A'] = \Bigg[&\sum_{a,b} (ab\,w^2)\, \langle\langle F_a\rangle \langle F_b\rangle\rangle_{k=1} \\
&- \left(\sum_a (aw)\, \langle\langle F_a\rangle\rangle_{k=1} \right)^2 \Bigg] \left\langle \frac{1}{\sum_i k_i} \right\rangle \\
&+ \left[\sum_a (a\,w)\, \langle\langle F_a\rangle\rangle - \tilde{x} \right]^2
\end{aligned}$$

and hence almost purely geometric when the 2nd term dominates in the R.H.S.

[B’] is given by

$$[B'] := \sum_{a,b} (ab\, w^2) \left\langle \frac{\sum_i k_i \left(\langle\langle F_a\, F_b \rangle\rangle_{k_i} - \langle\langle F_a \rangle \langle F_b \rangle\rangle_{k_i} \right)}{\left(\sum_i k_i\right)^2} \right\rangle$$

where

$$\langle\langle F_a\, F_b \rangle\rangle_k := \frac{1}{\bar{\bar{P}}(k)} \int \frac{dy}{\Delta Y} \bar{P}_{SI}(k;y) \int d\Delta x\, P_D(\Delta x; \sigma_d) \\ \times\; F_a(\tilde{x} + y \tan\phi + \Delta x)\, F_b(\tilde{x} + y \tan\phi + \Delta x)$$

This term represents the contributions from diffusion, gas gain fluctuation, and finite pad pitch. In the de-clustering limit, [B’] becomes

$$\lim_{\frac{\sigma_d}{L} \to \infty} [B'] := \sum_{a,b} (ab\, w^2) \left[\langle\langle F_a\, F_b \rangle\rangle_{k=1} - \langle\langle F_a \rangle \langle F_b \rangle\rangle_{k=1} \right] \\ \times \left\langle \frac{1}{\sum_i k_i} \right\rangle$$

The last term [C] is exactly as before and from the electronic noise contribution

$$[C] := \left(\frac{\sigma_E}{\bar{G}} \right)^2 \left\langle \frac{1}{\left(\sum_i k_i\right)^2} \right\rangle \sum_a (a\, w)^2$$

As far as the clustering effect is negligible this term is independent of the drift length as before.

As noted before, the de-clustering limit is never reached in practice, since the pad row height is so chosen. It is hence useful to consider an asymptotic formula in such a case where the diffusion is large enough compared to the pad width while it is short enough compared to the pad height.