Extended Kalman Filter

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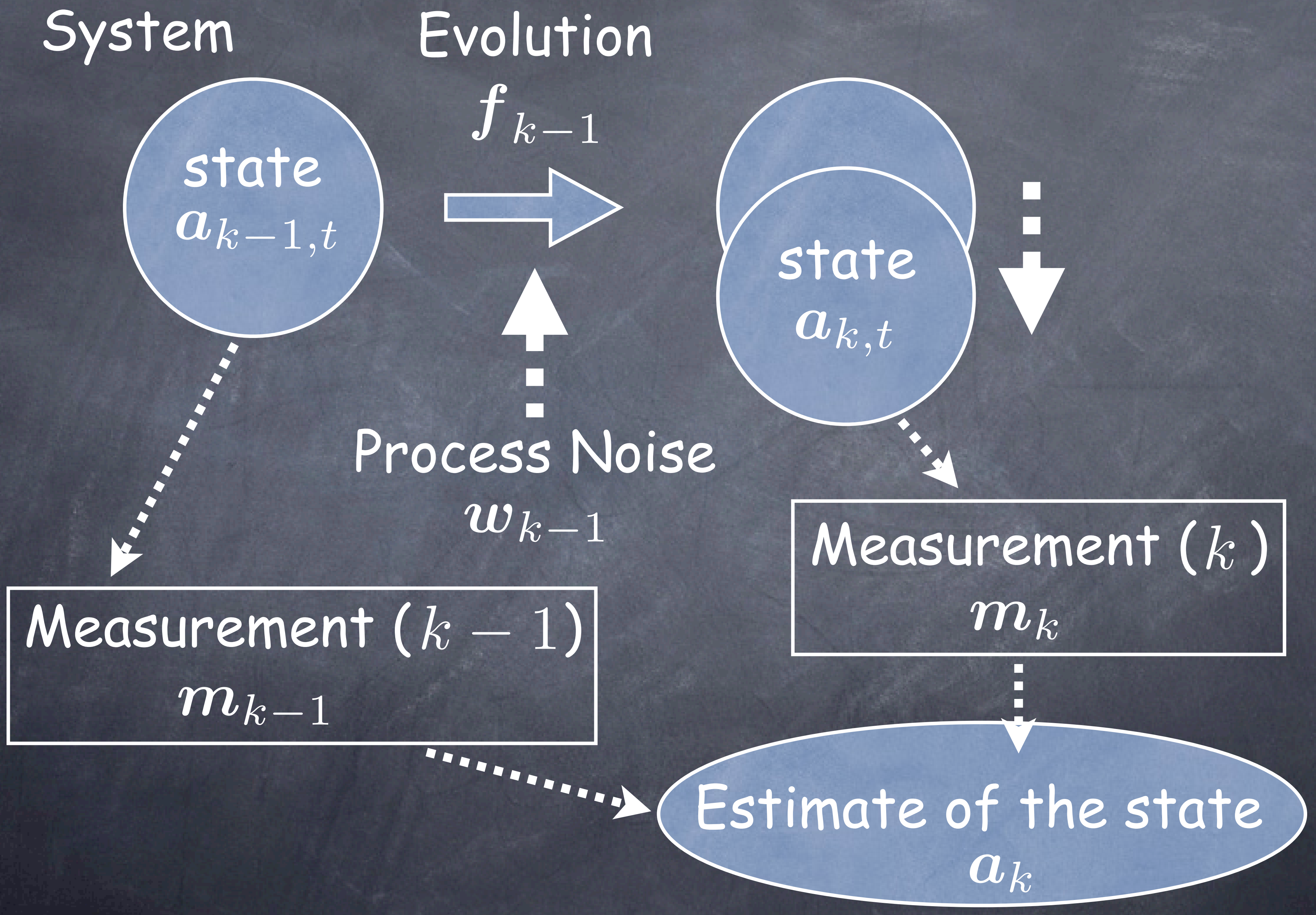
Statement of the Problem


System
Evolution
$\boldsymbol{f}_{k-1}$
state
$\boldsymbol{a}_{k-1,t}$
state
$\boldsymbol{a}_{k,t}$
Process Noise
$\boldsymbol{w}_{k-1}$
Measurement (k)
$\boldsymbol{m}_k$
Measurement ($k-1$)
$\boldsymbol{m}_{k-1}$
Estimate of the state
$\boldsymbol{a}_k$

System Equation (Equation of Motion)

$$\boldsymbol{a}_{k,t} = \boldsymbol{f}_{k-1}(\boldsymbol{a}_{k-1,t}) + \boldsymbol{w}_{k-1}$$

process noise from $(k-1)$ to (k)

true state vector at (k)

true state vector at (k)

Assume that process noise is random and unbiased

$$\begin{cases} \langle \boldsymbol{w}_k \rangle = \boldsymbol{0} \\ \langle \boldsymbol{w}_k \boldsymbol{w}_k^T \rangle \equiv \boldsymbol{Q}_k \end{cases}$$

Measurement Equation

$$\boldsymbol{m}_k = \boldsymbol{h}_k(\boldsymbol{a}_{k,t}) + \boldsymbol{\epsilon}_k$$

measurement noise

true measurement vector at $(k-1)$

measurement vector at (k)

Assume that measurement noise is random and unbiased

$$\begin{cases} \langle \boldsymbol{\epsilon}_k \rangle & = & \mathbf{0} \\ \left\langle \boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^T \right\rangle & \equiv & \boldsymbol{V}_k \equiv \boldsymbol{G}_k^{-1} \end{cases}$$

Example 1 : Ballistic Missile (Original Application)

$$\boldsymbol{a}_k = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{pmatrix}_k$$

Position and momentum at (k)

$$\boldsymbol{w}_{k-1}$$

Random turbulence between $(k-1)$ and (k)

$$\boldsymbol{m}_k$$

position and velocity measured with a radar at (k)

$$\boldsymbol{\epsilon}_k$$

Measurement error of radar

Example 2 : Tracking in HEP Experiments

$$\boldsymbol{a}_k = \begin{pmatrix} d_\rho \\ \phi_0 \\ \kappa \\ d_z \\ \tan\lambda \end{pmatrix}_k$$

Helix parameter vector at (k)

$\boldsymbol{w}_{k-1}$

Multiple scattering between $(k-1)$ and (k)

$\boldsymbol{m}_k$

Measured hit point at (k)

$\boldsymbol{\epsilon}_k$

random detector noise

Notation

$$\left\{\begin{array}{lll} \boldsymbol{a}_k^i & : & \text{extimate of } \boldsymbol{a}_{k,t} \text{ using measurements up to } (i) \\ & & \qquad (\boldsymbol{a}_k^k \equiv \boldsymbol{a}_k \text{ for simplicity of notation}) \\ \boldsymbol{C}_k^i & : & \text{covariance matrix for } \boldsymbol{a}_k^i \\ & & \qquad \boldsymbol{C}_k^i \equiv \left\langle (\boldsymbol{a}_k^i - \boldsymbol{a}_{k,t})(\boldsymbol{a}_k^i - \boldsymbol{a}_{k,t})^T \right\rangle \\ \boldsymbol{r}_k^i & : & \text{residual} \\ & & \qquad \boldsymbol{r}_k^i \equiv \boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^i) \\ \boldsymbol{R}_k^i & : & \text{covariance matrix for } \boldsymbol{r}_k^i \\ & & \qquad \boldsymbol{R}_k^i \equiv \left\langle \boldsymbol{r}_k^i \boldsymbol{r}_k^{iT} \right\rangle \end{array}\right.$$

What We Need = Recurrence Formulae

Machineary to do:

(i) Prediction

$$\{\boldsymbol{m}_{k'}; k' \leq k\} \mapsto \boldsymbol{a}_{k''>k} \qquad : \text{ future}$$

(ii) Filtering

$$\{\boldsymbol{m}_{k'}; k' \leq k\} \mapsto \boldsymbol{a}_{k''=k} \qquad : \text{ present}$$

(iii) Smoothing

$$\{\boldsymbol{m}_{k'}; k' \leq k\} \mapsto \boldsymbol{a}_{k''<k} \qquad : \text{ past}$$

Prediction

$$\{\boldsymbol{m}_{k'}; k' \leq k\} \mapsto \boldsymbol{a}_{k''>k} \quad : \text{ future}$$

- **State Vector**

$$\boldsymbol{a}_k^{k-1} = \boldsymbol{f}_{k-1}(\boldsymbol{a}_{k-1})$$

- **Covariance Matrix**

Extrapolation Error

Process Noise

$$\boldsymbol{C}_k^{k-1} = \boldsymbol{F}_{k-1}\boldsymbol{C}_{k-1}\boldsymbol{F}_{k-1}^T + \boldsymbol{Q}_{k-1}$$

$$\boldsymbol{F}_{k-1} \equiv \left(\frac{\partial \boldsymbol{f}_{k-1}}{\partial \boldsymbol{a}_{k-1}}\right)$$

- Residual

$$\boldsymbol{r}_k^{k-1} \equiv \boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^{k-1})$$

- Covariance Matrix

Extrapolation Error

$$\boldsymbol{R}_k^{k-1} = \boldsymbol{V}_k + \boldsymbol{H}_k \boldsymbol{C}_k^{k-1} \boldsymbol{H}_k^T$$

Measurement Noise

$$\boldsymbol{H}_k \equiv \left(\frac{\partial \boldsymbol{h}_k}{\partial \boldsymbol{a}_k^{k-1}} \right)$$

Filtering

$$\{\boldsymbol{m}_{k'}; k' \leq k\} \mapsto \boldsymbol{a}_{k''=k} \quad : \text{ present}$$

- State Vector

New Pull

$$\boldsymbol{a}_k = \boldsymbol{a}_k^{k-1} + \boldsymbol{K}_k \left(\boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^{k-1}) \right)$$

Kalman Gain Matrix

$$\begin{aligned} \boldsymbol{K}_k &\equiv \boldsymbol{C}_k^{k-1} \boldsymbol{H}_k^T \left(\boldsymbol{V}_k + \boldsymbol{H}_k \boldsymbol{C}_k^{k-1} \boldsymbol{H}_k^T \right)^{-1} \\ &= \boldsymbol{C}_k^{k-1} \boldsymbol{H}_k^T \left(\boldsymbol{R}_k^{k-1} \right)^{-1} \end{aligned}$$

already calculated in the prediction step

Covariance Matrix

$$\boldsymbol{C}_k = (\mathbf{1} - \boldsymbol{K}_k \boldsymbol{H}_k)\, \boldsymbol{C}_k^{k-1}$$

Equivalent but different Way: Weighted Mean Method

$$\boldsymbol{C}_k = \left[\left(\boldsymbol{C}_k^{k-1} \right)^{-1} + \boldsymbol{H}_k^T \boldsymbol{G}_k \boldsymbol{H}_k \right]^{-1}$$

$$\boldsymbol{K}_k = \boldsymbol{C}_k \boldsymbol{H}_k^T \boldsymbol{G}_k$$

Improvement from New Measurement at (k)

Which to use depends on the dimensions of state vector and measurement vector

- Residual

$$\boldsymbol{r}_k \equiv \boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k)$$
$$= (\boldsymbol{1} - \boldsymbol{H}_k \boldsymbol{K}_k)\, \boldsymbol{r}_k^{k-1}$$

- Covariance Matrix

$$\boldsymbol{R}_k = (\boldsymbol{1} - \boldsymbol{H}_k \boldsymbol{K}_k)\, \boldsymbol{V}$$
$$= \boldsymbol{V}_k - \boldsymbol{H}_k \boldsymbol{C}_k \boldsymbol{H}_k^T$$

Measurement Noise

Gain due to Information from previous measurements

- Chi Square Increment

$$\chi^2_+ = \boldsymbol{r}_k^T \boldsymbol{R}_k^{-1} \boldsymbol{r}_k$$
$$= \boldsymbol{r}_k^T \boldsymbol{G}_k \boldsymbol{r}_k + (\boldsymbol{a}_k - \boldsymbol{a}_k^{k-1})^T \left(\boldsymbol{C}_k^{k-1}\right)^{-1} (\boldsymbol{a}_k - \boldsymbol{a}_k^{k-1})$$

Smoothing

$$\{\boldsymbol{m}_{k'}; k' \leq k\} \mapsto \boldsymbol{a}_{k''<k} \quad : \text{ past}$$

- State Vector

New Pull

$$\boldsymbol{a}_k^n = \boldsymbol{a}_k + \boldsymbol{A}_k(\boldsymbol{a}_{k+1}^n - \boldsymbol{a}_{k+1}^k)$$

Smoothing Matrix

$$\boldsymbol{A}_k \equiv \boldsymbol{C}_k \boldsymbol{F}_k^T \left(\boldsymbol{C}_{k+1}^k\right)^{-1}$$

already calculated in the prediction step

already calculated in the prediction step

already calculated in the filtering step

It is instructive to compare filtering and smoothing formulae

Filter

$$\boldsymbol{a}_k = \boldsymbol{a}_k^{k-1} + \boldsymbol{K}_k \left(\boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^{k-1})\right)$$

$$\boldsymbol{K}_k \equiv \boldsymbol{C}_k^{k-1} \boldsymbol{H}_k^T \left(\boldsymbol{R}_k^{k-1}\right)^{-1}$$

New Pull

Smoothing

$$\boldsymbol{A}_k \equiv \boldsymbol{C}_k \boldsymbol{F}_k^T \left(\boldsymbol{C}_{k+1}^k\right)^{-1}$$

$$\boldsymbol{a}_k^n = \boldsymbol{a}_k + \boldsymbol{A}_k (\boldsymbol{a}_{k+1}^n - \boldsymbol{a}_{k+1}^k)$$

Covariance Matrix

Improvement from
Measurements at $(k+1 \sim n)$

negative definite

$$\boldsymbol{C}_k^n = \boldsymbol{C}_k + \boldsymbol{A}_k \left(\boldsymbol{C}_{k+1}^n - \boldsymbol{C}_{k+1}^k \right) \boldsymbol{A}_k^T$$

already calculated in the prediction step

already calculated in the prediction step

already calculated in the filtering step

- Residual

$$\boldsymbol{r}_k^n \equiv \boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^n)$$
$$= \boldsymbol{r}_k - \boldsymbol{H}_k(\boldsymbol{a}_k^n - \boldsymbol{a}_k)$$

- Covariance Matrix

$$\boldsymbol{R}_k^n = \boldsymbol{R}_k - \boldsymbol{H}_k \boldsymbol{A}_k \left(\boldsymbol{C}_{k+1}^n - \boldsymbol{C}_{k+1}^k \right) \boldsymbol{A}_k^T \boldsymbol{H}_k^T$$
$$= \boldsymbol{V}_k - \boldsymbol{H}_k \boldsymbol{C}_k^n \boldsymbol{H}_k^T$$

Measurement Noise ($\boldsymbol{V}_k$)

Gain due to Information from other measurements ($\boldsymbol{H}_k \boldsymbol{C}_k^n \boldsymbol{H}_k^T$)

Inverse Kalman Filter

Machineary to eliminate measurement (k)

- State Vector

Pull we want to eliminate

$$\boldsymbol{a}_k^{n*} = \boldsymbol{a}_k^n + \boldsymbol{K}_k^{n*} \left(\boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^n) \right)$$

Kalman Inverse Gain Matrix

already calculated in the prediction step

$$\boldsymbol{K}_k^{n*} \equiv \boldsymbol{C}_k^n \boldsymbol{H}_k^T \left(-\boldsymbol{V}_k + \boldsymbol{H}_k \boldsymbol{C}_k^n \boldsymbol{H}_k^T \right)^{-1}$$

already calculated in the smoothing step

Covariance Matrix for state vector

$$\boldsymbol{C}_k^{n*} = (\boldsymbol{1} - \boldsymbol{K}_k^{n*}\boldsymbol{H}_k)\,\boldsymbol{C}_k^n$$
$$= \left[(\boldsymbol{C}_k^n)^{-1} - \boldsymbol{H}_k^T \boldsymbol{G}_k \boldsymbol{H}_k \right]^{-1}$$

already calculated in the prediction step

already calculated in the smoothing step

Covariance Matrix for residual

$$\boldsymbol{R}_k^{n*} = \boldsymbol{V}_k + \boldsymbol{H}_k \boldsymbol{C}_k^{n*} \boldsymbol{H}_k^T$$

Summary

Typical Procedure for Tracking

outermost — innermost

$$(1) \rightarrow (2) \rightarrow \cdots \rightarrow (k-1) \rightarrow (k) \rightarrow \cdots (n)$$

filtering

$$\boldsymbol{a}_1 \nearrow \boldsymbol{a}_2^1 \downarrow \boldsymbol{a}_2 \nearrow \cdots \nearrow \boldsymbol{a}_{k-1}^{k-2} \downarrow \boldsymbol{a}_{k-1} \nearrow \boldsymbol{a}_k^{k-1} \downarrow \boldsymbol{a}_k \nearrow \cdots \nearrow \boldsymbol{a}_n$$

prediction

$$(1) \cdots \leftarrow (k) \leftarrow (k+1) \leftarrow \cdots \leftarrow (n-1) \leftarrow (n)$$

$$\boldsymbol{a}_1^n \cdots \Leftarrow \boldsymbol{a}_k^n \Leftarrow \boldsymbol{a}_{k+1}^n \Leftarrow \cdots \Leftarrow \boldsymbol{a}_{n-1}^n \Leftarrow \boldsymbol{a}_n \equiv \boldsymbol{a}_n^n$$

smoothing

Extrapolation

$$\boldsymbol{a}_n \Rightarrow \boldsymbol{a}_{n+1}^n = \boldsymbol{a}_{IP}$$

prediction

$$\boldsymbol{a}_0^n = \boldsymbol{a}_{\text{cal}} \Leftarrow \boldsymbol{a}_1^n$$

prediction

Alignment, Resolution Study, etc.

Need to eliminate point (k)

$$(1)\cdots\cdots\cdots(k-1) \quad (k) \quad (k+1)\cdots\cdots\cdots(n)$$

↓ Inverse Kalman Filter

$\boldsymbol{a}_k^{n*}$ Reference Track Param.

↓

$\boldsymbol{h}_k(\boldsymbol{a}_k^{n*})$ Expected Hit Position

↓

$\boldsymbol{r}_k^{n*} = \boldsymbol{m}_k - \boldsymbol{h}_k(\boldsymbol{a}_k^{n*})$ Residual to Look At