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# Introduction to Helical Track Manipulations

JLC Physics Group
KEK, Tsukuba 305, Japan

**Abstract**

Techniques for helical track manipulations are reviewed in an introductory fashion, aiming at applications to collider experiments equipped with tracking detectors in a uniform magnetic field. The topics include numerical treatments of track fitting, track extrapolation, track linking, and vertexing, as well as an analytic approach valid in high momentum limit.

# Chapter 1

# Introduction

Manipulations of charged particle tracks in a uniform magnetic field are unavoidable part of everyday life of experimentalists working on collider experiment data. We usually fit set of hit points in a tracking device to a helix to determine the momentum of the track. We sometimes need to extrapolate it through some materials to connect it to the corresponding track segment detected in another tracking device, taking into account energy loss and multiple scattering. Tracks reconstructed this way are then combined to form primary and secondary vertices, and so on. These tasks often involve complicated numerical procedures such as track parameter transformations and error propagations. It is therefore highly desirable to formulate the problems in unified and consistent manner.

In this review, we present an introduction to helix manipulation techniques in as unified form as possible. We start with the helix parametrization of a track exploited throughout this review together with some comments on generalities of track fitting. We then discuss the extrapolation of the track to other detector regions through material media, where the track is expected to experience energy loss and multiple scattering. This enables us to link track segments reconstructed in different regions of a detector system to a single track, which will be our next topic. Vertexing of so reconstructed tracks follows it together with techniques to implement geometrical constraints. The results up to this point assume that original helix track segments are given numerically together with their error matrices and are applicable to any helix track regardless of its momentum. It is, however, useful to consider high momentum limit, since it allows us to calculate error matrices analytically and thus enables us to estimate the performance of a tracking device analytically. This will be the last topic of our review.

# Chapter 2

# General Discussions without Momentum Restriction

## 2.1 Helix Parametrization

A charged particle in a uniform magnetic field follows a helical trajectory. In what follows we set our reference frame so that it has its $z$ axis in the direction of the magnetic field. There are many ways to parametrize this helix. The following choice, however, greatly facilitates various operations we are going to make on tracks, such as track extrapolation and linking:

$$\left\{ \begin{array}{lllllll} x &=& x_0 &+& d_\rho \cos\phi_0 &+& \frac{\alpha}{\kappa}\left(\cos\phi_0 - \cos(\phi_0+\phi)\right) \\ y &=& y_0 &+& d_\rho \sin\phi_0 &+& \frac{\alpha}{\kappa}\left(\sin\phi_0 - \sin(\phi_0+\phi)\right) \\ z &=& z_0 &+& d_z &-& \frac{\alpha}{\kappa}\tan\lambda\cdot\phi, \end{array} \right. \tag{2.1.1}$$

where $\mathbf{x}_0 = (x_0, y_0, z_0)^T$ specifies an arbitrarily chosen pivotal point or simply pivot: we usually take as a pivot a hit or wire position of the tracking device or a position of track linking. Once the pivotal point is fixed, the helix is determined by a 5-component parameter vector $\mathbf{a} = (d_\rho, \phi_0, \kappa, d_z, \tan\lambda)^T$, where $d_\rho$ is the distance of the helix from the pivotal point in the $xy$ plane, $\phi_0$ is the azimuthal angle to specifies the pivotal point with respect to the helix center, $\kappa$ is the signed reciprocal transverse momentum so that

$$\begin{array}{lll} \kappa &=& Q/P_T \\ \rho &=& \alpha/\kappa \end{array} \tag{2.1.2}$$

with $Q$ being the charge, $\rho$ being the signed radius of the helix, and $\alpha \equiv 1/cB$ being a magnetic-field-dependent constant, $d_z$ is the distrance of the helix from the pivotal point in the $z$ direction, and $\tan\lambda$ is the dip angle. The deflection angle $\phi$ is measured from the pivotal point and specifies the position of the charged particle on the helical track. The meanings of these parameters are depicted in Figs.2.1-a) for negatively charged and -b) for positively charged particles.[1]

[1]The vectors in Fig.2.1 are defined by

$$\vec{X} = \vec{X}_0 + (\rho + d_\rho)\cdot\vec{W} - \rho\cdot\vec{V}$$

Notice that a negatively charged particle travels in the increasing $\phi$ direction, while a positively charged particle travels in the decreasing $\phi$ direction.

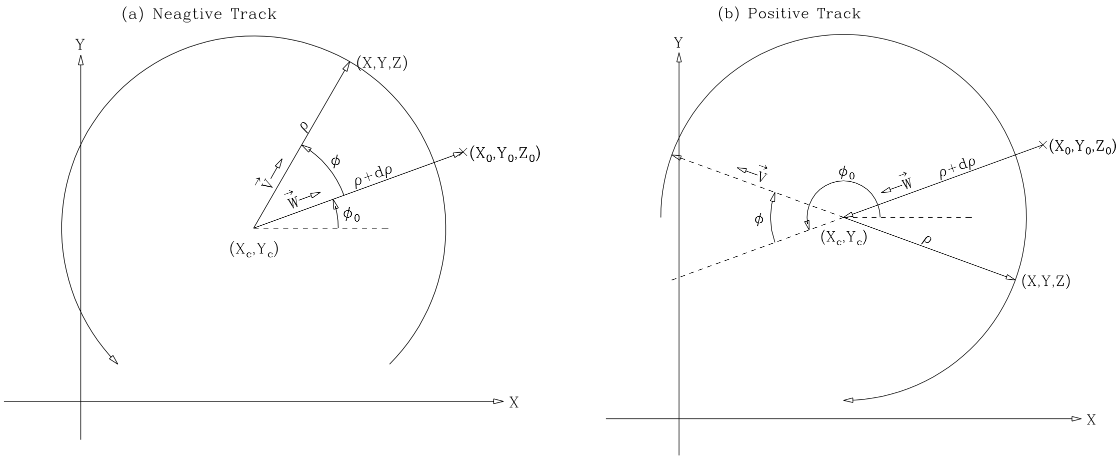


Figure 2.1: A graphical explanation of helix parameters for (a) negatively charged and (b) positively charged tracks. Notice that the meaning of $\phi_0$ changes discretely by $\pi$, depending on the charge.

Taking this into account it is easy to get the following formula which relates the momentum of a charged particle to its helix parameters:

$$\mathbf{p} = -\left(\frac{Q}{\alpha}\right)\frac{d\mathbf{x}}{d\phi} = \frac{1}{|\kappa|}\begin{pmatrix} -\sin(\phi_0+\phi) \\ \cos(\phi_0+\phi) \\ \tan\lambda \end{pmatrix}. \qquad (2.1.3)$$

## 2.2 Generalities of Track Fitting

Track fitting is the procedure to determine helix parameters by fitting a set of coordinates measured in a tracking detector to a helix. The $\chi^2$ to minimize has, in general, the following form:

$$\chi^2 = \sum_{i=1}^{n}\left(\frac{\xi_i - \xi(i,\mathbf{a})}{\sigma_i}\right)^2 \qquad (2.2.4)$$

where $\xi_i$ is $i$-th measured coordinate, $\sigma_i$ is its error, and $\xi(i,\mathbf{a})$ is the expected $i$-th coordinate, when the helix parameter vector is $\mathbf{a}$. What we need is the helix parameter vector which zeros the first derivative of $\chi^2$:

$$\frac{\partial\chi^2}{\partial\mathbf{a}} = 2\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\Delta_i\left(\frac{\partial\Delta_i}{\partial\mathbf{a}}\right), \qquad (2.2.5)$$

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$$\begin{aligned}\vec{W} &= (\cos\phi_0, \sin\phi_0)^T \\ \vec{V} &= (\cos(\phi_0+\phi), \sin(\phi_0+\phi))^T.\end{aligned}$$

where we have defined the $i$-th residual $\Delta_i$ by

$$\Delta_i = \xi_i - \xi(i, \mathbf{a}). \tag{2.2.6}$$

This can be numerically found by iteratively from the following equation (multi-dimensional Newton's method):

$$\mathbf{a}_{\nu+1} = \mathbf{a}_\nu - \left(\frac{\partial^2 \chi^2}{\partial \mathbf{a}^T \partial \mathbf{a}}\right)_\nu^{-1} \cdot \left(\frac{\partial \chi^2}{\partial \mathbf{a}^T}\right)_\nu, \tag{2.2.7}$$

where the left-hand side is the $(\nu+1)$-th estimate of the helix parameter vector based on the knowledge on the right-hand side of its $\nu$-th estimate together with the first and the second derivatives of the $\chi^2$ thereat. The second derivative is given by

$$\begin{aligned} \frac{\partial^2 \chi^2}{\partial \mathbf{a}^T \partial \mathbf{a}} &= 2\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left[\left(\frac{\partial \Delta_i}{\partial \mathbf{a}^T}\right) \cdot \left(\frac{\partial \Delta_i}{\partial \mathbf{a}}\right) + \Delta_i \left(\frac{\partial^2 \Delta_i}{\partial \mathbf{a}^T \partial \mathbf{a}}\right)\right] \\ &\simeq 2\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left(\frac{\partial \Delta_i}{\partial \mathbf{a}^T}\right) \cdot \left(\frac{\partial \Delta_i}{\partial \mathbf{a}}\right), \end{aligned} \tag{2.2.8}$$

where in the last line we have deliberately left out the second derivatives of $\Delta_i$'s in order to make positive definite the second derivative matrix of $\chi^2$[2]. When the initial estimate of the parameter vector $\mathbf{a}_0$ is a good approximation, the parameter vector converges rapidly after a few times of iterations. Nevertheless, it is usually recommended to multiply the diagonal elements of the second derivative matrix by some constant which is greater than unity and try again, when the $\chi^2$ increased for the new estimate of the parameter vector. This prescription renders the parameter change vector along the opposite gradient direction in such cases and stabilizes the fit.

When the fit converges, the error matrix for the parameter vector is obtained to be

$$E_{\mathbf{a}} = \left(\frac{1}{2} \frac{\partial^2 \chi^2}{\partial \mathbf{a}^T \partial \mathbf{a}}\right)^{-1}. \tag{2.2.9}$$

It should be emphasized that we have to carefully choose the helix parametrization so as to numerically stabilize the fitting procedure: the helix parameters should stay small and continuously change during the fit. Notice that our parametrization allows a continuous transition from a negative charge solution to a positive charge solution:

$$\kappa < 0 \to \kappa = 0 \to \kappa > 0$$

with other parameters stay the same. Since this transition implies that the center of the helix jumps from a point infinitely away on one side of the track to another infinitely away point on the other side of the track, it changes the meaning of $\phi_0$ discretely by $\pi$.

[2] The terms containing the second derivatives of $\Delta_i$'s are proportional to $\Delta_i$ and are hence small and usually negligible near the $\chi^2$ minimum.

## 2.3 Change of Pivot

As it will become clear, it is very useful to establish procedure to change pivot positions, since an appropriately chosen pivotal points simplifies calculations necessary, for instance, in energy loss or multiple scattering corrections significantly.

The pivot change

$$\mathbf{x}_0 = (x_0, y_0, z_0)^T \rightarrow \mathbf{x}'_0 = (x'_0, y'_0, z'_0)^T$$

induces the following change in the helix parameter vector:

$$\mathbf{a} = (d_\rho, \phi_0, \kappa, d_z, \tan\lambda)^T \rightarrow \mathbf{a}' = (d'_\rho, \phi'_0, \kappa', d'_z, \tan\lambda')^T,$$

where the new parameters are given in terms of the old ones as follows:

$$\begin{array}{lcl} d'_\rho & = & \left(x_0 - x'_0 + \left(d_\rho + \frac{\alpha}{\kappa}\right)\cos\phi_0\right)\cos\phi'_0 \\ & & + \left(y_0 - y'_0 + \left(d_\rho + \frac{\alpha}{\kappa}\right)\sin\phi_0\right)\sin\phi'_0 - \frac{\alpha}{\kappa} \\ \phi'_0 & = & \arctan\left(\frac{y_0 - y'_0 + \left(d_\rho + \frac{\alpha}{\kappa}\right)\sin\phi_0}{x_0 - x'_0 + \left(d_\rho + \frac{\alpha}{\kappa}\right)\cos\phi_0}\right) + \frac{\pi}{2}(1 + Q/|Q|) \\ \kappa' & = & \kappa \\ d'_z & = & z_0 - z'_0 + d_z - \left(\frac{\alpha}{\kappa}\right)\cdot(\phi'_0 - \phi_0)\cdot\tan\lambda \\ \tan\lambda' & = & \tan\lambda. \end{array} \tag{2.3.10}$$

The above relations can be readily obtained by requiring the primed parameter vector represents the same helix as the unprimed one. The error matrix should also be transformed accordingly:

$$E_{\mathbf{a}'} = \left(\frac{\partial \mathbf{a}'}{\partial \mathbf{a}}\right) \cdot E_{\mathbf{a}} \cdot \left(\frac{\partial \mathbf{a}'}{\partial \mathbf{a}}\right)^T, \tag{2.3.11}$$

where the calculation of the transformation (Jacobian) matrix is straightforward from Eq.2.3.10 but it is rather tedious and therefore not shown here.

## 2.4 Track Extrapolation

In this section, we will show how to extrapolate a helical track through a material to the other detector region where the magnetic field is not necessarily present. For simplicity, we assume that the energy loss and the multiple scattering in the material take place at a single point in space[3]: $\mathbf{x} = \mathbf{x}_{int}$. The track extrapolation is easiest when this interaction point is chosen as the pivot:

$$\mathbf{x}_0 = \mathbf{x}_{int} = \mathbf{x}(\phi = 0, \mathbf{a}; \mathbf{x}_0).$$

In what follows, it shoud be understood that the track parameters and error matrrix are pre-transformed to this pivot, according to the procedure explained in the last section, unless otherwise stated.

[3]This assumption is valid for a thin material. If it is not, we can always slice the material into thin enough sublayers and apply the method explained here repeatedly. For a high momentum track, a different treatment is possible as described later.

### 2.4.a Energy Loss Correction

When the pivot is chosen at the energy loss point, the only track parameter that is subject to the energy loss correction is the curvature:

$$\kappa' = \kappa + \Delta\kappa_{dE/dx}, \tag{2.4.12}$$

where the second term on the right-hand side is the correction calculated from the average energy loss in the material[4]. This implies

$$\left(\frac{\partial \mathbf{a}'}{\partial \mathbf{a}}\right) = 1, \tag{2.4.13}$$

therefore, the error matrix remains the same.

Notice that, unless the pivot is chosen at the energy loss point, all the helix parameters but $\tan\lambda$ are affected by the energy loss, which demonstrates the advantage of having the freedom to arbitrarily choose the pivotal position in our helix parametrization.

### 2.4.b Multiple Scattering Correction

Unless we have an extra-tracking device in the new region, we do not know the scattering angle in the material. The best we can do is, therefore, to take into account the effect of the multiple scattering on the error matrix.

It can be shown (see Appendix A) that the error matrix for the track extended through the scatterer becomes

$$E_{\mathbf{a}'} = E_{\mathbf{a}} + E_{MS}, \tag{2.4.14}$$

where the second term on the right-hand side represents the correction to the error matrix due to the multiple scattering:

$$\begin{aligned}
(E_{MS})_{22} &= \sigma_{MS}^2 \cdot (1 + \tan^2\lambda) \\
(E_{MS})_{33} &= \sigma_{MS}^2 \cdot (\kappa\tan\lambda)^2 \\
(E_{MS})_{35} &= \sigma_{MS}^2 \cdot \kappa\tan\lambda(1 + \tan^2\lambda) \\
(E_{MS})_{55} &= \sigma_{MS}^2 \cdot (1 + \tan^2\lambda)^2
\end{aligned} \tag{2.4.15}$$

with all the other components being zero. The $\sigma_{MS}$ is given, as usual, by

$$\sigma_{MS} = \frac{0.0141}{P(\mathrm{GeV})\beta}\sqrt{X_L}\left(1 + \frac{1}{9}\log_{10} X_L\right), \tag{2.4.16}$$

where $P$, $\beta$, and $X_L$ are the momentum, the velocity in units of the light velocity, and the thickness of the scatterer in units of its radiation length.

In summary, we can extrapolate a helical track through a material by first moving the pivot to the intersection of the track and the material and then making the aformentioned

[4] We assume that the fluctuation of the energy loss is negligible in what follows.

modifications to the track parameters and error matrix. If there are two or more scatterers, all we have to do is just repeat this process.

Once the error matrix for the extrapolated track is calculated, it is easy, for instance, to estimate the position error at a deflection angle $\phi$ from the error matrix given by

$$E_{\mathbf{x}}(\phi) = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{a}'}\right) \cdot E_{\mathbf{a}'} \cdot \left(\frac{\partial \mathbf{x}}{\partial \mathbf{a}'}\right)^T . \tag{2.4.17}$$

## 2.4.c Continuation to Straight Line

Let us first consider the extrapolation of a track in a uniform magnetic field $\mathbf{B}$ to an adjacent region of a different magnetic field $\mathbf{B}'$:

$$\alpha(\mathbf{B}) \to \alpha(\mathbf{B}').$$

If the pivot is taken at the intersection of the track with the boundary of the two regions, the helix parameter vector is left unchanged so is the error matrix, except for the energy loss and multiple scattering correctoins at the boundary.

The zero field case corresponds to the infinite $\alpha'$ limit, which implies that $\phi$ goes to zero while $\alpha'\phi$ is kept finite. Our helix parametrization allows us to take this limit easily. We expand the helix equations (Eq.2.1.1) in terms of the deflection angle $\phi$ to the first order:

$$\left\{ \begin{array}{ccccccccl} x & = & x_0 & + & d_\rho \cos\phi_0 & + & \left(-\frac{\alpha'\phi}{\kappa}\right) & \cdot & (-\sin\phi_0) \\ y & = & x_0 & + & d_\rho \sin\phi_0 & + & \left(-\frac{\alpha'\phi}{\kappa}\right) & \cdot & (\quad \cos\phi_0) \\ z & = & z_0 & + & d_z & - & \left(-\frac{\alpha'\phi}{\kappa}\right) & \cdot & (\quad \tan\lambda\ ). \end{array} \right. \tag{2.4.18}$$

These become our straight line parametrization, when we make the replacement:

$$\left(-\frac{\alpha'\phi}{\kappa}\right) \to \text{const.} \equiv t, \tag{2.4.19}$$

where the straight line parameter vector is given by

$$\mathbf{b} = (d_\rho, \phi_0, d_z, \tan\lambda)^T \tag{2.4.20}$$

and their error matrix is obtained by deleting the third row and the third column of the error matrix for the helix:

$$E_{\mathbf{b}} = \left(\frac{\partial \mathbf{b}}{\partial \mathbf{a}'}\right) \cdot E_{\mathbf{a}'} \cdot \left(\frac{\partial \mathbf{b}}{\partial \mathbf{a}'}\right)^T \tag{2.4.21}$$

with

$$\left(\frac{\partial \mathbf{b}}{\partial \mathbf{a}'}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.4.22}$$

## 2.5 Track Linking

There are two ways to link tracks to improve the precision of track parameter measurements. The first method is to combine track parameters and their error matrices obtained by individual tracking devices. In the second method, we go all way back to hit points and refit them to a single track consisting of several track segments. Apparently the second is more powerful, since it uses all of the available information. Unlike the first method, this, however, heavily depends on the nature of the coordinate information from each tracking divice. Thus, we will review the first method here.

### 2.5.a Helix to Helix

Before combining track segments, we move their pivots to a common point which can be arbitrarily chosen. We should also correct the track parameters for energy loss and multiple scattering beforehand so that they represent the track parameters for the original track. Then the total $\chi^2$ for the combined track is then given by

$$\chi^2 = \sum_i \Delta\mathbf{a}_i^T \cdot E_{\mathbf{a}_i}^{-1} \cdot \Delta\mathbf{a}_i, \qquad (2.5.23)$$

where

$$\Delta\mathbf{a}_i = \mathbf{a} - \mathbf{a}_i$$

and $\mathbf{a}$ is the track parameter vector for the original track, $\mathbf{a}_i$ is that determined by $i$-th tracking device, and $E_{\mathbf{a}_i}$ is the corresponding error matrix. Since this is quadratic in $\mathbf{a}$, the minimization of the $\chi^2$ can be carried out analytically by a single matrix inversion:

$$\begin{aligned} \overline{\mathbf{a}} &= \left(\sum_i E_{\mathbf{a}_i}^{-1}\right)^{-1} \cdot \left(\sum_i E_{\mathbf{a}_i}^{-1} \cdot \mathbf{a}_i\right) \\ \overline{E}_{\mathbf{a}} &= \left(\sum_i E_{\mathbf{a}_i}^{-1}\right)^{-1}. \end{aligned} \qquad (2.5.24)$$

The above formulae tell us that the track linking in the parameter space is a simple averaging process[5].

### 2.5.b Helix to Straight Line

Given a straight line track and a helical track, we can combine them to improve the helix parameter measurements as we did for the helix to helix linking[6]. Again the pivots should

[5] Notice that the helix parameter vector so obtained is the one for the original track before energy loss and multiple scattering, which can then be used for vertexing. It should be also noted that the track segments combined here may belong to detector regions with different magnetic fields, as long as the fields can be regarded as constant in each region.

[6] Since any straight line can be regarded as the zero field limit of some helix, the results shown below is a special case of the results in the last subsection

be chosen at a common space point: the best choice is the intersection of the straight line track with the boundary of the two regions. We then correct the error matrix of the straight line track for multiple scattering. Then the $\chi^2$ to minimize is defined by

$$\chi^2 = \Delta\mathbf{a}_H^T \cdot E_{\mathbf{a}_H}^{-1} \cdot \Delta\mathbf{a}_H + \Delta\mathbf{a}_S^T \cdot E_{\mathbf{a}_S}^{-1} \cdot \Delta\mathbf{a}_S \tag{2.5.25}$$

with

$$\begin{aligned} \Delta\mathbf{a}_H &= \mathbf{a}_H - \mathbf{a} \\ \Delta\mathbf{a}_S &= \mathbf{a}_S - \mathbf{a}, \end{aligned}$$

where $\mathbf{a}_H$ and $E_{\mathbf{a}_H}$ are the helix parameter vector and its error matrix, while $\mathbf{a}_S$ and $E_{\mathbf{a}_S}$ are derived from those of the straight line ($\mathbf{b}$ and $E_\mathbf{b}$) as follows[7]:

$$\begin{aligned} &\mathbf{b} = (b_1, b_2, b_3, b_4)^T \quad \rightarrow \quad \mathbf{a}_S = (b_1, b_2, 0, b_3, b_4)^T \\ &E_\mathbf{b}^{-1} = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \quad \rightarrow \quad E_{\mathbf{a}_S}^{-1} = \begin{pmatrix} & & 0 & & \\ & * & & & * \\ & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \\ & & 0 & & \\ & * & & & * \\ & & 0 & & \end{pmatrix}. \end{aligned} \tag{2.5.26}$$

Then what follows is the same as with the helix to helix linking in the previous subsection: it reduces to the two-track case: $i = H,\ S$ in Eq.2.5.24[8].

## 2.6 Vertex Fitting

In this section, we discuss the way to find the common vertex of a given set of helical tracks. We first consider a simple geometrical approach which can be used even if the error matrices are unknown or unreliable. When the error matrices are known, statistically more appropriate treatments are possible. We extend our simple method without error matrices to the case with error matrices and then consider the way to include geometrical constraints.

### 2.6.a Vertex Fitting without Error Matrix

In order to supply an initial vertex position, we first calculate an approximate vertex position by calculating the intersection in the $xy$ plane of two helices arbitrarily chosen from the given set of tracks. There are two such intersections in general. We then compares the differences of $z$ coordinates and take the intersection corresponding to the smaller distance. The mid-point in the $z$ direction is our first guess of the common vertex

[7] The $*$'s in Eq.2.5.26 represent $2 \times 2$ matrices.

[8] It is easy to generalize the results to the cases in which two or more regions of no magnetic field are present.

$\mathbf{x}_v = (x_v, y_v, z_v)^T$. We approximate the helices to tangential lines at the points on the helices that are the closest in the $xy$ plane to the trial vertex position:

$$\mathbf{x} = \mathbf{x}_v^i + t\ \mathbf{e}_i \tag{2.6.27}$$

where

$$\begin{aligned} \mathbf{x}_v^i &= \mathbf{x}(\phi_i, \mathbf{a}_i) && : \text{the point on } i\text{-th helix closest in the } xy \text{ plane to the trial vertex point } \mathbf{x}_v \\ \mathbf{e}_i &= \frac{\partial \mathbf{x}}{\partial \phi_i} \Big/ \left| \frac{\partial \mathbf{x}}{\partial \phi_i} \right| && : \text{the unit tangential vector to the } i\text{-th helix at } \mathbf{x}_v^i \end{aligned} \tag{2.6.28}$$

and $\phi_i$ is given by

$$\phi_i = \arctan\left(\frac{y_v - y_c^i}{x_v - x_c^i}\right) - \phi_0^i + \frac{\pi}{2}(1 + Q/|Q|) \tag{2.6.29}$$

with $(x_c^i, y_c^i)$ being the center of the $i$-th helix:

$$\begin{aligned} x_c^i &= x_0^i + \left(d_\rho^i + \frac{\alpha}{\kappa^i}\right) \cos \phi_0^i \\ y_c^i &= y_0^i + \left(d_\rho^i + \frac{\alpha}{\kappa^i}\right) \sin \phi_0^i. \end{aligned} \tag{2.6.30}$$

Then, what to minimize is the sum of the distance squared of each tangential line from a new trial vertex $\mathbf{x'}_v$:

$$\chi^2 = \sum_i d_i^2 \tag{2.6.31}$$

with

$$d_i^2(\mathbf{x'}_v) = (\mathbf{x'}_v - \mathbf{x}_v^i)^2 - \left(\mathbf{e}_i \cdot (\mathbf{x'}_v - \mathbf{x}_v^i)\right)^2. \tag{2.6.32}$$

Notice that the $\chi^2$ is a quadratic funciton of $\mathbf{x'}_v$ so that the minimization condition

$$\frac{\partial \chi^2}{\partial \mathbf{x'}_v} = 0 \tag{2.6.33}$$

is a linear equation which can be solved by a single matrix inversion. The solution $\mathbf{x'}_v$ is our improved guess of the common vertex. We repeat this process until the vertex position converges.

### 2.6.b Vertex Fitting with Error Matrix

When error matrices are available, the $\chi^2$ to minimize is defined by

$$\chi^2 = \sum_i \Delta\mathbf{x}_i^T \cdot E_{\mathbf{x}_i}^{-1} \cdot \Delta\mathbf{x}_i \tag{2.6.34}$$

where

$$\Delta\mathbf{x}_i = \mathbf{x}_v - \mathbf{x}(\phi_i; \mathbf{a}_i)$$

and $E_{\mathbf{x}_i}$ is the position error matrix of $i$-th track at the deflection angle $\phi = \phi_i$ corresponding to the trial vertex $\mathbf{x}_v$:

$$E_{\mathbf{x}_i} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{a}_i}\right) \cdot E_{\mathbf{a_i}} \cdot \left(\frac{\partial \mathbf{x}}{\partial \mathbf{a}_i}\right)^T .$$

The parameter vector to determine by $\chi^2$ minimization is

$$\mathbf{A} \equiv \begin{pmatrix} \mathbf{x}_v \\ \boldsymbol{\phi} \end{pmatrix}$$

where $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_i, \cdots, \phi_{n_{track}})$ and thus $\mathbf{A}$ contains $3 + n_{n_{track}}$ parameters. The $\chi^2$ minimization requires the calculations of the first and the second derivatives of the $\chi^2$. The first derivatives are calculated as

$$\begin{aligned}
\frac{\partial \chi^2}{\partial \mathbf{x}_v} &= 2 \sum_i \Delta\mathbf{x}_i^T \cdot E_{\mathbf{x}_i}^{-1} \\
\frac{\partial \chi^2}{\partial \phi_i} &= -2 \sum_i \left(\frac{\partial \mathbf{x}}{\partial \phi_i}\right)^T \cdot E_{\mathbf{x}_i}^{-1} \cdot \Delta\mathbf{x}_i \\
&\qquad - \Delta\mathbf{x}_i^T \cdot E_{\mathbf{x}_i}^{-1} \cdot \left(\frac{\partial E_{\mathbf{x}_i}}{\partial \phi_i}\right) \cdot E_{\mathbf{x}_i}^{-1} \cdot \Delta\mathbf{x}_i \\
&\simeq -2 \sum_i \left(\frac{\partial \mathbf{x}}{\partial \phi_i}\right)^T \cdot E_{\mathbf{x}_i}^{-1} \cdot \Delta\mathbf{x}_i,
\end{aligned} \tag{2.6.35}$$

where in the last step we have ignored the second order term with respect to the residual vector $\Delta\mathbf{x}_i$ which is a good approximation near the $\chi^2$ minimum. From these the second derivatives are readily obtained:

$$\begin{aligned}
\frac{\partial^2 \chi^2}{\partial \mathbf{x}_v^T \partial \mathbf{x}_v} &= 2 \sum_i E_{\mathbf{x}_i}^{-1} \\
\frac{\partial^2 \chi^2}{\partial \phi_i \partial \mathbf{x}_v} &\simeq -2 \sum_i \left(\frac{\partial \mathbf{x}}{\partial \phi_i}\right)^T \cdot E_{\mathbf{x}_i}^{-1} \\
\frac{\partial^2 \chi^2}{\partial \phi_i \partial \phi_j} &\simeq -2 \sum_i \left(\frac{\partial \mathbf{x}}{\partial \phi_i}\right)^T \cdot E_{\mathbf{x}_i}^{-1} \cdot \left(\frac{\partial \mathbf{x}}{\partial \phi_i}\right) \delta_{ij}.
\end{aligned} \tag{2.6.36}$$

Notice that when the position error matrices are unity, the problem is equivalent to the one in the last subsection. The difference here is the introduction of the error matrices as metric tensors in the residual space.

Once the derivatives are calculated, all we have to do is to solve the following linear equation (multi-dimensonal Newton's method) iteratively:

$$\mathbf{A}_{\nu+1} = \mathbf{A}_\nu - \left(\frac{\partial^2 \chi^2}{\partial \mathbf{A}^T \partial \mathbf{A}}\right)_\nu^{-1} \cdot \left(\frac{\partial \chi^2}{\partial \mathbf{A}^T}\right)_\nu , \tag{2.6.37}$$

where the diagonal elements of the second derivative matrix should be multiplied by some number greater than one when the $\chi^2$ increases as with the track fitting in Section 2.2.

When the fit converges, the error matrix for the parameter vector is obtained to be

$$E_{\mathbf{A}} = \left( \frac{1}{2} \frac{\partial^2 \chi^2}{\partial \mathbf{A}^T \partial \mathbf{A}} \right)^{-1} . \tag{2.6.38}$$

## 2.6.c Inclusion of Geometrical Constraints

In the last subsection, we tried to determine the common vertex of given $n_{track}$ tracks without touching the track parameter vectors themselves: these helical tracks do not necessarily pass through the common vertex. Here, we require these tracks to originate from a common vertex:

$$\mathbf{x}_v = (x_v, y_v, z_v)^T$$

so that $i$-th track, for instance, can be parametrized as

$$\left\{ \begin{array}{lllll} x & = & x_v & + & \frac{\alpha}{\tilde{\kappa}^i} \left( \cos \tilde{\phi}_0^i - \cos(\tilde{\phi}_0^i + \phi) \right) \\ y & = & y_v & + & \frac{\alpha}{\tilde{\kappa}^i} \left( \sin \tilde{\phi}_0^i - \sin(\tilde{\phi}_0^i + \phi) \right) \\ z & = & z_v & - & \frac{\alpha}{\tilde{\kappa}^i} \tan \tilde{\lambda}^i \cdot \phi . \end{array} \right. \tag{2.6.39}$$

Notice that, for a given trial vertex $\mathbf{x}_v$, the track parameter vector has now only three components:

$$\tilde{\mathbf{a}}_i = (\tilde{\phi}_0^i, \tilde{\kappa}^i, \tan \tilde{\lambda}^i)^T,$$

since the track has to pass through the pivot $\mathbf{x}_v$ and therefore $\tilde{d}_\rho = \tilde{d}_z = 0$.

In order to calculate the $\chi^2$, we need to transform the pivot of the $i$-th track from the trial vertex $\mathbf{x}_v$ to the pivot of the corresponding measured track. This induces the following change of the helix parameters:

$$\begin{array}{lll} d_\rho^i & = & \left( x_v - x_0^i + \left( \frac{\alpha}{\tilde{\kappa}^i} \right) \cos \tilde{\phi}_0^i \right) \cos \phi_0^i \\ & & + \left( y_v - y_0^i + \left( \frac{\alpha}{\tilde{\kappa}^i} \right) \sin \tilde{\phi}_0^i \right) \sin \phi_0^i - \frac{\alpha}{\tilde{\kappa}^i} \\ \phi_0^i & = & \arctan \left( \frac{y_v - y_0^i + \left( \frac{\alpha}{\tilde{\kappa}^i} \right) \sin \tilde{\phi}_0^i}{x_v - x_0^i + \left( \frac{\alpha}{\tilde{\kappa}^i} \right) \cos \tilde{\phi}_0^i} \right) + \frac{\pi}{2} (1 + Q^i / |Q^i|) \\ \kappa^i & = & \tilde{\kappa}^i \\ d_z^i & = & z_v - z_0^i - \left( \frac{\alpha}{\tilde{\kappa}^i} \right) \cdot (\phi_0^i - \tilde{\phi}_0^i) \cdot \tan \tilde{\lambda}^i \\ \tan \lambda^i & = & \tan \tilde{\lambda}^i , \end{array} \tag{2.6.40}$$

which gives the helix parameter vector:

$$\mathbf{a}(\tilde{\mathbf{a}}^i, \mathbf{x}_v) = (d_\rho^i, \phi_0^i, \kappa^i, d_z^i, \tan \lambda^i)^T$$

to be compared with the corresponding measured track parameter vector $\mathbf{a}_a$. Thus, we arrive at the $\chi^2$ definition:

$$\chi^2 = \sum_i \Delta \mathbf{a}_i^T \cdot E_{\mathbf{a}_i}^{-1} \cdot \Delta \mathbf{a}_i \tag{2.6.41}$$

with

$$\mathbf{a}_i = \mathbf{a}_i - \mathbf{a}(\tilde{\mathbf{a}}^i, \mathbf{x}_v),$$

the minimization of which determinies the parameter vector:

$$\mathbf{A} \equiv \begin{pmatrix} \mathbf{x}_v \\ \tilde{\mathbf{a}}_1 \\ \vdots \\ \tilde{\mathbf{a}}_i \\ \vdots \\ \tilde{\mathbf{a}}_{n_{track}} \end{pmatrix}$$

containing $3 \cdot (n_{track} + 1)$ components. Notice that the necessary calculations of the first and the second derivatives of the $\chi^2$ require only the evaluation of the transformation matrix:

$$\frac{\partial \mathbf{a}(\tilde{\mathbf{a}}^i, \mathbf{x}_v)}{\partial \mathbf{A}}$$

and what then follows is the same as with the last subsection.

# Chapter 3

# High Momentum Limit

In this chapter, we examine the simplifications that take place in the high momentum limit and their possible applications to detector design considerations.

## 3.1 Track Model

We consider a high momentum track originating from the interaction point (IP) region. When we take our local coordinate system in such a way that the $z$ axis is along the magnetic field, the $y$ axis in the radial direction which is approximately along the transverse momentum of the track, and the $x$ axis makes the overall system right-handed, the helical track can be approximated by[1]

$$\left\{ \begin{array}{lllllll} x & \simeq & x_0 & + & \xi & - & \eta(y-y_0) + \frac{1}{2}\left(\frac{\alpha}{\kappa}\right)(y-y_0)^2 \\ z & \simeq & z_0 & + & \zeta - \xi\eta a & + & a(y-y_0) \\ & \simeq & z_0 & + & \zeta & + & a(y-y_0), \end{array} \right. \tag{3.1.1}$$

where we have introduced the following short-hand for the helix parameters:

$$\begin{array}{lll} \xi & \equiv & d_\rho \\ \eta & \equiv & \phi_0 \\ \zeta & \equiv & d_z \\ a & \equiv & \tan\lambda. \end{array}$$

Notice that the last approximation ($|\xi\eta| \ll |y-y_0|$), which is justified by our choice of the coordinate system ($r \leftrightarrow y \Rightarrow \eta \equiv \phi_0 \simeq 0$), makes the $r$-$\phi$ and the $r$-$z$ fittings decouple from each other. We will, thus, treat the $r$-$\phi$ and the $r$-$z$ fittings separately in what follows. Notice also that the problem then becomes a linear one, which simplifies the necessary calculations considerably.

[1] In Subection 2.4.c, we have Taylor-expand Eq.2.1.1 to the lowest order to get a straight line track. The track model here, on the other hand, corresponds to the next order approximation.

## 3.2 Track Fitting in $r$-$\phi$ Plane

In this section, we will first focus our attention on the tracking in the $r$-$\phi$ plane.

### 3.2.a Pivot Transformation in Vacuum

We will show below that in the high momentum limit the effects of multiple scattering can be implemented in a continuous manner. For this purpose, it turns out very useful to set up machinary to move the pivot, taking into account the effects of multiple scattering.

We first consider the pivot transformation without multiple scattering. The track parameter vector ($\mathbf{a} = (\xi, \eta, \kappa)^T$) transforms under an infinitesimal change of pivot:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \to \begin{pmatrix} x_0 + dx_0 \\ y_0 + dy_0 \end{pmatrix}$$

as follows:

$$\begin{pmatrix} \xi \\ \eta \\ \kappa \end{pmatrix} \to \begin{pmatrix} \xi & - & dx_0 - \eta\ dy_0 \\ \eta & - & \left(\frac{\kappa}{\alpha}\right) dy_0 \\ \kappa & & \end{pmatrix}.$$

This can be cast into a vector equation:

$$d\mathbf{a} = -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dx_0 - T \cdot \mathbf{a}\ dy_0 \tag{3.2.2}$$

where

$$T \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/\alpha \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.2.3}$$

From now on, we keep $dx_0 = 0$[2], which reduces the pivot transformation to a single parameter group and the above equation to

$$\frac{d\mathbf{a}}{dy_0} = -T \cdot \mathbf{a}. \tag{3.2.4}$$

Under this condition, $x_0 + \xi(y_0)$ represents the $x$ coordinate of the track at $y = y_0$. In terms of the generator $T$, the transformation matrix for the error matrix is given by

$$\left(\frac{\partial \mathbf{a}'(y_0 + dy_0)}{\partial \mathbf{a}(y_0)}\right) = 1 - T\ dy_0 \tag{3.2.5}$$

which implies

$$dE = -\left(T \cdot E + E \cdot T^T\right)\ dy_0. \tag{3.2.6}$$

[2]This does not spoil the generality of our treatment, since a nonzero $\Delta x_0$ can always recovered as a simple shift in $\xi$.

### 3.2.b Inclusion of Multiple Scattering

When the pivot is taken at the point of multiple scattering, only $\eta$ is affected among the track parameters $(\xi, \eta, \kappa)$[3]. The scattering angle in the $r$-$\phi$ plane $\Delta\eta \equiv \Delta\phi_0$ has a variance

$$\begin{aligned} \sigma^2_{\Delta\phi_0} &= (1+\tan^2\lambda)\ \sigma^2_{MS} \\ &\simeq (1+a^2)\left(\frac{C}{P\beta}\right)^2 \frac{(1+a^2)^{1/2}|\Delta y_0|}{X_0} \equiv K\ |\Delta y_0| \end{aligned} \quad (3.2.7)$$

with $C = 0.0141$GeV, $P$ and $\beta$ are the momentum and the velocity of the particle, and $X_0$ is the radiation length of the material. This modifies $(E)_{22}$ and Eq.3.2.6 now becomes

$$dE = -\left(T \cdot E + E \cdot T^T\right)\ dy_0 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{pmatrix} |dy_0|. \quad (3.2.8)$$

Notice that $(E)_{33} \equiv \sigma^2_\kappa$ is not affected by the pivot transformation, which allows us to readily integrate the above equation[4]. The solution of the above equation can be written in the form:

$$E = E_G + E_S$$

where $E_G$ is the general solution for $K = 0$, while $E_S$ is a special solution of Eq.3.2.8. It is easily verified that

$$E_S = \begin{pmatrix} \frac{K}{3}|\Delta y_0|^3 & -\frac{K}{2}|\Delta y_0|^2 & 0 \\ -\frac{K}{2}|\Delta y_0|^2 & K|\Delta y_0| & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.2.9)$$

satisfies Eq.3.2.8[5]. It is also easy to show that

$$E_G = \left(\frac{\partial \mathbf{a}}{\partial \mathbf{a}_0}\right) \cdot E_0 \cdot \left(\frac{\partial \mathbf{a}}{\partial \mathbf{a}_0}\right)^T \quad (3.2.10)$$

with

$$\left(\frac{\partial \mathbf{a}}{\partial \mathbf{a}_0}\right) \equiv A = \begin{pmatrix} 1 & -\Delta y_0 & -\frac{1}{2\alpha}\Delta y_0^2 \\ 0 & 1 & -\frac{1}{\alpha}\Delta y_0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.2.11)$$

---

[3] We will ignore the small effects on $\kappa \equiv Q/P_T$ due to the change in the dip angle $a \equiv \tan\lambda$:

$$\Delta\kappa = \frac{a\ \Delta a}{1+a^2}\kappa \simeq 0,$$

which approximation is justifiable when $a \lesssim 1$.

[4] In practice $K$ is usually region-dependent. We should, therefore, integrate this equation segment by segment.

[5] $\Delta y_0 \equiv y_0' - y_0$ is now a finite shift of the $y$ coordinate of the pivot.

is the general solution of the homogeneous equation ($K = 0$)[6].

In summary, the propagated error matrix at the point which is outside the area of coordinate measurements is the sum[7]:

$$E(\Delta y_0) = E_G(\Delta y_0) + E_S(\Delta y_0), \tag{3.2.12}$$

where $E_G$ is the propagated error matrix without taking multiple scattering into account (Eqs.3.2.10 and 3.2.11) and $E_S$ is the contribution from multiple scattering (Eq.3.2.9).

Now that we have established the machinary to move the pivot to anywhere in the detector system, it is easy to implement the information from external tracking devices. Assume that we have an extra coordinate measurement $x = \bar{x}$ at $y = \bar{y}$. Then we take $(0, \bar{y})$ as our new pivot and minimize

$$\chi^2 = (\mathbf{a}' - \bar{\mathbf{a}})^T \cdot E^{-1}(\bar{y}) \cdot (\mathbf{a}' - \bar{\mathbf{a}}) + \left(\frac{\xi - \bar{x}}{\sigma_{\bar{x}}}\right)^2, \tag{3.2.13}$$

which is a simple matrix inversion. The corresponding error matrix can also be readily obtained from

$$E'^{-1} = \frac{1}{2}\frac{\partial^2\chi^2}{\partial\mathbf{a}'^T\partial\mathbf{a}'} = E^{-1}(\bar{y}) + \begin{pmatrix} 1/\sigma_{\bar{x}}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.2.14}$$

If there are more than one extra measurements, we can retransform $E'$ to the next and do the same.

If $\sigma_{\bar{x}}$ is negligibly small, we might even include this as a constraint on the track parameter. In this case, the track becomes

$$x = \bar{x} - \eta(y - \bar{y}) + \frac{1}{2}\left(\frac{\kappa}{\alpha}\right)(y - \bar{y})^2 \tag{3.2.15}$$

which has only two parameters:

$$\mathbf{A} \equiv (\eta, \kappa)^T.$$

[6]In fact, we have

$$\begin{aligned} \frac{d}{dy_0}E_G &= \left(\frac{dA}{dy_0}\right) \cdot E_0 \cdot A^T + A \cdot E_0 \cdot \left(\frac{dA}{dy_0}\right)^T \\ &= \left[\left(\frac{dA}{dy_0}\right) \cdot A^{-1}\right] \cdot E_G + E_G \cdot \left[\left(\frac{dA}{dy_0}\right) \cdot A^{-1}\right]^T, \end{aligned}$$

while we know

$$A(\Delta y_0 + dy_0) \cdot A^{-1}(\Delta y_0) = A(dy_0) \quad \rightarrow \quad \frac{d}{dy_0}A = -T \cdot A \quad \rightarrow \quad \left(\frac{dA}{dy_0}\right) \cdot A^{-1} = -T$$

from Eq.3.2.5.

[7]If the new pivot is within the sensitive volume of the tracker which measured the track in question, we should drop the $E_S$ term, since the multiple scattering effects in the tracker must presumably be included in the original error matrix.

The error matrix for $\mathbf{A}$ is given by

$$E_{\mathbf{A}} = \left[ \left( \frac{\partial \mathbf{a}}{\partial \mathbf{A}} \right)^T \cdot E^{-1}(\bar{y}) \cdot \left( \frac{\partial \mathbf{a}}{\partial \mathbf{A}} \right) \right]^{-1} \tag{3.2.16}$$

with

$$\left( \frac{\partial \mathbf{a}}{\partial \mathbf{A}} \right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2.17}$$

### 3.2.c Track Fitting under the Influence of Multiple Scattering

When the track is affected by multiple scattering during its coordinate measurements the error matrix ($E_0$ in Eq.3.2.10) should take that into account. To see how to implement multiple scattering, let us start with the modification of the track equation (Eq.3.2.4):

$$\frac{d\mathbf{a}}{dy} = -T \cdot \mathbf{a},$$

where we have set $(x_0, y_0) = (0, 0)$ and $y \equiv \Delta y_0$. Multiple scattering modifies $\eta$ by

$$d\eta \equiv d\phi_0 = \left( \frac{d\phi}{dy} \right) \, dy \tag{3.2.18}$$

which necessitates the modification of the track equation[8] to

$$\frac{d\mathbf{a}}{dy} = -T \cdot \mathbf{a} + \begin{pmatrix} 0 \\ \frac{d\phi}{dy} \\ 0 \end{pmatrix}. \tag{3.2.19}$$

This means

$$\left\{ \begin{array}{rclcccl} \xi(y) & = & \xi_0 & - & \eta_0 y & + & \left(\frac{\kappa_0}{\alpha}\right) y^2 + \int^y \phi(y) dy \\ \eta(y) & = & \eta_0 & - & \left(\frac{\kappa_0}{\alpha}\right) y & + & \phi(y) \\ \kappa(y) & = & \kappa_0, & & & & \end{array} \right. \tag{3.2.20}$$

where the suffix 0 attached to the track track parameters remines us that they are the track parameters at the starting point before any multiple scattering. In the pivot convention taken here $(x_0, y_0) = (0, 0)$, $\xi(y)$ is the $x$-location of the track. Therefore, the probability of observing $(n + 1)$ hit points, $(x_i, y_i); i = 0, 1, \cdots, n$ and $y_0 = 0, \cdots, y_n = L$, is

$$P_M(\mathbf{x}; \phi; \mathbf{a}_0) = N \ \exp \left[ -\frac{1}{2} \sum_{i=1}^{n} \left( \frac{x_i - \xi(y_i; \phi; \mathbf{a}_0)}{\sigma_{x_i}} \right)^2 \right], \tag{3.2.21}$$

[8] The functional form of $\phi(y)$ is unknown so that it must be integrated out eventually.

where $N$ is a normalization factor and $\mathbf{x} \equiv (x_0, \cdots, x_n)^T$. We must, however, take into account the probability of obtaining the multiple scattering $\phi(y)$:

$$\begin{aligned} P_{MS}(\phi; \mathbf{a}_0) &= N' \exp\left[ -\frac{1}{2} \sum \left( \frac{\frac{d\phi}{dy^\nu} \Delta y^\nu}{\sigma_{\Delta\phi_0^\nu}} \right)^2 \right] \\ &= N' \exp\left[ -\frac{1}{2} \int_0^L \frac{1}{K} \left( \frac{d\phi}{dy} \right)^2 dy \right], \end{aligned} \tag{3.2.22}$$

where use has been made of Eq.3.2.7. Since we do not know $\phi(y)$, we should integrate the probability over $\phi(y)$:

$$\begin{aligned} \overline{P}(\mathbf{x}; \mathbf{a}_0) &= \int [d\phi] \; P_M(\mathbf{x}; \phi; \mathbf{a}_0) \cdot P_{MS}(\phi; \mathbf{a}_0) \\ &= N'' \int [d\phi] \; \exp\left[ -\frac{1}{2} \int_0^L \frac{1}{K} \left( \frac{d\phi}{dy} \right)^2 dy \right] \\ &\qquad \cdot \exp\left[ -\frac{1}{2} \sum_{i=1}^n \left( \frac{x_i - \xi(y_i; \phi; \mathbf{a}_0)}{\sigma_{x_i}} \right)^2 \right], \end{aligned} \tag{3.2.23}$$

which now gives the probability of observing $\mathbf{x} = (x_0, \cdots, x_n)$ when the track has the track parameter $\mathbf{a}_0$ at $y = 0$.

Now, we fit the $(n+1)$ hits to our track model (Eq.3.1.1) by minimizing

$$\chi^2 = \sum_{i=0}^n \left( \frac{x_i - x(y_i, \mathbf{a})}{\sigma_{x_i}} \right)^2 \tag{3.2.24}$$

where $x(y, \mathbf{a})$ is given by Eq.3.1.1 with $x_0 = 0$. Since this problem is linear, we can easily obtain

$$\mathbf{a} = E_M \cdot \sum_{i=0}^n \frac{x_i}{\sigma_{x_i}^2} \begin{pmatrix} 1 \\ -y_i \\ \frac{y_i^2}{2\alpha} \end{pmatrix} \tag{3.2.25}$$

with

$$E_M^{-1} = \begin{pmatrix} \sum \frac{1}{\sigma_{x_i}^2} & -\sum \frac{y_i}{\sigma_{x_i}^2} & \left(\frac{1}{2\alpha}\right) \sum \frac{y_i^2}{\sigma_{x_i}^2} \\ & \sum \frac{y_i^2}{\sigma_{x_i}^2} & -\left(\frac{1}{2\alpha}\right) \sum \frac{y_i^3}{\sigma_{x_i}^2} \\ & & \left(\frac{1}{2\alpha}\right)^2 \sum \frac{y_i^4}{\sigma_{x_i}^2} \end{pmatrix}, \tag{3.2.26}$$

where $E_M^{-1}$ is the inverse of the error matrix due to corrdinate measurement errors, for which only the upper triangle is explicitly shown[9]. Of course, the true error matrix is

[9]Notice that, in the high momentum limit where the track can be approximated by a parabola, $E_M$ is determined solely by the configuration of the tracking detector in question: the $y$-locations of sampling points and the spatial resolutions thereat.

subsect to the correction for multiple scattering. Let us now derive the true error matrix below. We first note that the displacement of $\mathbf{x}$ induces the displacement of $\mathbf{a}$ through Eq.3.2.25:

$$
\begin{aligned}
\Delta\mathbf{a} &= E_M \cdot \begin{pmatrix} 1 & \cdots & 1 \\ -y_0 & \cdots & -y_n \\ \frac{y_0^2}{2\alpha} & \cdots & \frac{y_n^2}{2\alpha} \end{pmatrix} \cdot \begin{pmatrix} \Delta x_0/\sigma_{x_0}^2 \\ \vdots \\ \Delta x_n/\sigma_{x_n}^2 \end{pmatrix} \\
&= E_M \cdot \begin{pmatrix} 1 & \cdots & 1 \\ -y_0 & \cdots & -y_n \\ \frac{y_0^2}{2\alpha} & \cdots & \frac{y_n^2}{2\alpha} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sigma_{x_0}^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_{x_1}^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\sigma_{x_n}^2} \end{pmatrix} \cdot \begin{pmatrix} \Delta x_0 \\ \vdots \\ \Delta x_n \end{pmatrix}. \qquad (3.2.27)
\end{aligned}
$$

Defining $A$ and $E_{\mathbf{x}}$ by

$$
A \equiv \begin{pmatrix} 1 & \cdots & 1 \\ -y_0 & \cdots & -y_n \\ \frac{y_0^2}{2\alpha} & \cdots & \frac{y_n^2}{2\alpha} \end{pmatrix} \quad \text{and} \quad E_{\mathbf{x}} \equiv \begin{pmatrix} \sigma_{x_0}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{x_1}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{x_n}^2 \end{pmatrix}, \qquad (3.2.28)
$$

we can rewrite Eq.3.2.27 as

$$
\Delta\mathbf{a} = E_M \cdot A \cdot E_{\mathbf{x}}^{-1} \cdot \Delta\mathbf{x}. \qquad (3.2.29)
$$

This equation implies the full error matrix including multiple scattering to be given by

$$
\begin{aligned}
\left\langle \Delta\mathbf{a} \cdot \Delta\mathbf{a}^T \right\rangle &\equiv \int d\mathbf{x}\ \overline{P}(\mathbf{x}; \mathbf{a}_0)\ (\Delta\mathbf{a} \cdot \Delta\mathbf{a}^T) \\
&= \left\langle E_M \cdot A \cdot E_{\mathbf{x}}^{-1} \cdot \Delta\mathbf{x} \cdot \Delta\mathbf{x}^T \cdot E_{\mathbf{x}}^{-1} \cdot A^T \cdot E_M \right\rangle \\
&= E_M \cdot A \cdot E_{\mathbf{x}}^{-1} \cdot \left\langle \Delta\mathbf{x} \cdot \Delta\mathbf{x}^T \right\rangle \cdot E_{\mathbf{x}}^{-1} \cdot A^T \cdot E_M. \qquad (3.2.30)
\end{aligned}
$$

Notice that in the absence of multiple scattering, we have

$$
\left\langle \Delta\mathbf{x} \cdot \Delta\mathbf{x}^T \right\rangle = E_{\mathbf{x}}
$$

and, therefore,

$$
\begin{aligned}
\left\langle \Delta\mathbf{a} \cdot \Delta\mathbf{a}^T \right\rangle &= E_M \cdot \left( A \cdot E_{\mathbf{x}}^{-1} \cdot A^T \right) \cdot E_M \\
&= E_M \cdot E_M^{-1} \cdot E_M = E_M
\end{aligned}
$$

as it should be, where in the last line, we have used

$$
\left( A \cdot E_{\mathbf{x}}^{-1} \cdot A^T \right) = E_M^{-1} \qquad (3.2.31)
$$

which can be readily verified.

Our problem now reduces to how to evaluate $\left\langle \Delta\mathbf{x} \cdot \Delta\mathbf{x}^T \right\rangle$ when the multiple scattering correlates, for instance, $\Delta x_i$ to $\Delta x_j$. In order to calculate the correlation matrix, we first note that

$$\begin{array}{rcl} \Delta x_i & = & x_i - \xi(y_i;\phi=0;\mathbf{a}_0) \\ & = & x_i - \underbrace{\xi(y_i;\phi;\mathbf{a}_0)}_{\substack{\uparrow \\ \text{true trajectory}}} + \underbrace{\xi(y_i;\phi;\mathbf{a}_0) - \xi(y_i;\phi=0;\mathbf{a}_0)}_{\substack{\| \\ \int_0^{y_i}\phi(y)dy}}. \end{array} \tag{3.2.32}$$

Substituting this in Eq.3.2.23, we obtain the following formula for the probability of getting the measured points $\mathbf{x} = (x_0, x_1, \cdots, x_n)^T$ for a track whose track parameter vector at $y = y_0$ is $\mathbf{a}_0$:

$$\begin{array}{rcl} \overline{P}(\mathbf{x};\mathbf{a}_0) & = & N'' \int [d\phi] \, \exp\left[-\frac{1}{2}\int_0^L \frac{1}{K}\left(\frac{d\phi}{dy}\right)^2 dy\right] \\ & & \cdot \exp\left[-\frac{1}{2}\sum_{i=0}^{n}\left(\frac{\Delta x_i - \int_0^{y_i}\phi(y)dy}{\sigma_{x_i}}\right)^2\right], \end{array} \tag{3.2.33}$$

which leads us to

$$\begin{array}{rcl} \langle \Delta x_i \Delta x_j \rangle & \equiv & \int d\Delta\mathbf{x} \; \overline{P}(\mathbf{x};\mathbf{a}_0) \; (\Delta x_i \Delta x_j) \\ & = & N'' \int d\Delta\mathbf{x} \int [d\phi] \, \exp\left[-\frac{1}{2}\int_0^L \frac{1}{K}\left(\frac{d\phi}{dy}\right)^2 dy\right] \\ & & \cdot \exp\left[-\frac{1}{2}\sum_{k=0}^{n}\left(\frac{\Delta x_k - \int_0^{y_k}\phi(y)dy}{\sigma_{x_k}}\right)^2\right] \; (\Delta x_i \Delta x_j) \\ & = & N'' \int [d\phi] \, \exp\left[-\frac{1}{2}\int_0^L \frac{1}{K}\left(\frac{d\phi}{dy}\right)^2 dy\right] \\ & & \cdot \int d\Delta\mathbf{x}' \, \exp\left[-\frac{1}{2}\sum_{k=0}^{n}\left(\frac{\Delta x'_k}{\sigma_{x_k}}\right)^2\right] \\ & & \cdot \left(\Delta x'_i + \int_0^{y_i}\phi(y)dy\right)\left(\Delta x'_j + \int_0^{y_j}\phi(y)dy\right). \end{array}$$

The $\Delta\mathbf{x}'$ integral is readily performed to yield

$$\begin{array}{rcl} \langle \Delta x_i \Delta x_j \rangle & = & \delta_{ij}\sigma_{x_i}^2 + N' \int [d\phi] \, \exp\left[-\frac{1}{2}\int_0^L \frac{1}{K}\left(\frac{d\phi}{dy}\right)^2 dy\right] \\ & & \cdot \left(\int_0^{y_i}\phi(y)dy\right)\left(\int_0^{y_j}\phi(y)dy\right). \end{array} \tag{3.2.34}$$

The second term, which was induced by multiple scattering, has a familiar path integral form in field theories. After some straightforward manipulations such as to divide the path into small segments and to replace integrals by summentions and differentiations by differences, we arrive at

$$\text{the 2nd term} \equiv B_{ij} = \frac{K}{2} y_i^2 (y_j - y_i/3) \tag{3.2.35}$$

for $i \leq j$ which is enough since $B$ is apparently symmetric. Combining Eqs.3.2.30, 3.2.34, and 3.2.35, we obtain the full error matrix as the sum of the part due to coordinate measurement errors $E_M$ and the part due to the multiple scattering in the tracking volume $E_{MS}$:

$$E \equiv \left\langle \Delta\mathbf{a} \cdot \Delta\mathbf{a}^T \right\rangle = E_M + E_{MS} \tag{3.2.36}$$

with

$$E_{MS} = E_M \cdot A \cdot E_{\mathbf{x}}^{-1} \cdot B \cdot E_{\mathbf{x}}^{-1} \cdot A^T \cdot E_M. \tag{3.2.37}$$

The calculation of $E_{MS}$ is tedious but doable. We will, however, restrict ourselves to the case where the $(n+1)$ samples are equally spaced in $y$ and measured with an equal accuracy $\sigma_x$. We will further assume that $n$ is large, since the final expression otherwise becomes too complicated to show here[10]. The large $n$ limit then gives

$$E_M \simeq \sigma_x^2 \begin{pmatrix} \frac{9}{n} & \frac{36}{nL} & \frac{30\cdot(2\alpha)}{nL^2} \\ & \frac{192}{nL^2} & \frac{180\cdot(2\alpha)}{nL^3} \\ & & \frac{180\cdot(2\alpha)^2}{nL^4} \end{pmatrix} \tag{3.2.38}$$

and

$$E_{MS} \simeq K \begin{pmatrix} \frac{L^3}{630} & \frac{L^2}{60} & \frac{(2\alpha)L}{168} \\ & \frac{8L}{35} & \frac{3(2\alpha)}{28} \\ & & \frac{5(2\alpha)^2}{14L} \end{pmatrix}, \tag{3.2.39}$$

which, for instance, result in

$$\left(\sigma_\kappa^{MS}\right)^2 = (E_{MS})_{33} = \frac{10}{7} \cdot \frac{K\alpha^2}{L}.$$

The error on $\kappa \equiv Q/P_T$ due to multiple scattering is thus obtained to be

$$\begin{aligned} \left(\sigma_\kappa^{MS}\right)^2 &= \frac{10}{7} \cdot \frac{\alpha^2}{L} \cdot (1 + \tan^2\lambda)^{3/2} \left(\frac{C}{P\beta}\right)^2 \frac{1}{X_0} \\ &= \frac{10}{7} \cdot \left(\frac{\alpha}{L}\right)^2 \cdot \left(\frac{C}{P_T\beta}\right)^2 \frac{X}{X_0}, \end{aligned}$$

where use has been made of Eq.3.2.7 and

$$P_T^2 = P^2/(1 + \tan^2\lambda)$$

[10]Exact formulae for $E_M$ which are valid for any $n$ are given in Appendix B.

$$X \equiv (1+\tan^2\lambda)^{1/2}L.$$

The $X$ is the material thickness through which the track passes. To see the dependence on the magnetic field explicitly, we introduce a new constant $\alpha'$ defined by

$$\alpha' \equiv \alpha \cdot B.$$

When we assume $\beta = 1$ and $|Q| = 1$ (unit charge), we finally get a familiar result:

$$\sigma_\kappa^{MS}/\kappa = \frac{\alpha' C}{LB}\sqrt{\frac{10}{7}\left(\frac{X}{X_0}\right)} \tag{3.2.40}$$

with

$$\left\{ \begin{array}{lcl} \alpha' & = & 333.56\ (\mathrm{cm \cdot T \cdot GeV^{-1}}) \\ C & = & 0.0141\ (\mathrm{GeV}) \\ (X/X_0) & = & \text{thickness measured in radiation length units} \\ L & = & \text{lever arm length (cm)} \\ B & = & \text{magnetic field (T)} \end{array} \right. . \tag{3.2.41}$$

# 3.3 Track Fitting in $r$-$z$ Plane

Now that we have finished the $r$-$\phi$ fitting, let us move on to the $r$-$z$ fitting. The $r$-$z$ fitting can be treated exactly in the same way as with the $r$-$\phi$ case: in fact it is simpler. Therefore, we shall skip the details and only show results below.

## 3.3.a Pivot Transformation

The parameter vector for a $r$-$z$ track has only two components (see Eq.3.1.1): $\mathbf{a} \equiv (\zeta, a)^T$ which transforms as

$$d\mathbf{a} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} dz_0 + T \cdot \mathbf{a} dy_0,$$

where

$$T \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.3.42}$$

As we did in the $r$-$\phi$ case, we set $dz_0 = 0$. Then, we have, in vacuum,

$$\frac{d\mathbf{a}}{dy_0} = T \cdot \mathbf{a}. \tag{3.3.43}$$

The equation to transform the error matrix is then

$$dE = \left(T \cdot E + E \cdot T^T\right) dy_0 + K' \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} |dy_0| \tag{3.3.44}$$

with

$$\begin{aligned}\sigma_a^2 &= (1+\tan^2\lambda)^2\sigma_{MS}^2 \\ &\simeq (1+a^2)^2\left(\frac{C}{P\beta}\right)^2\frac{(1+a^2)^{1/2}|dy_0|}{X_0}.\end{aligned} \tag{3.3.45}$$

The solution to this equation can be written in the form:

$$E(\Delta y_0) = E_G(\Delta y_0) + E_S(\Delta y_0) \tag{3.3.46}$$

where $E_G$ is the general solution for $K' = 0$ and $E_S$ is a special solution:

$$E_G = \left(\frac{\partial \mathbf{a}}{\partial \mathbf{a}_0}\right)\cdot E_0 \cdot \left(\frac{\partial \mathbf{a}}{\partial \mathbf{a}_0}\right)^T \tag{3.3.47}$$

with

$$\left(\frac{\partial \mathbf{a}}{\partial \mathbf{a}_0}\right) = \begin{pmatrix} 1 & \Delta y_0 \\ 0 & 1 \end{pmatrix} \tag{3.3.48}$$

and

$$E_S = \begin{pmatrix} \frac{K'}{3}|\Delta y_0|^3 & \frac{K'}{2}|\Delta y_0|^2 \\ \frac{K'}{2}|\Delta y_0|^2 & K'|\Delta y_0| \end{pmatrix}, \tag{3.3.49}$$

where $E_0$ is the error matrix at $\Delta y_0 = 0$. Notice that the second term, which is due to multiple scattering, must be dropped when one moves the pivot in the sensitive volume of the tracker which gave $E_0$, since the effect of multiple scattering is presumably taken into account in $E_0$. For instance, if one wants to extrapolate the track in the increasing $y$ direction, one should first move the pivot to the outermost hit with $K' = 0$ and then use to above formulae.

## 3.3.b Track Fitting with Multiple Scattering

Now let us turn our attention to the $r$-$z$ track fitting under the influence of multiple scattering. The treatment here provides a way to calculate the $E_0$ of the last subsection including multiple scattering effects in track fitting. From now on, we take $(y_0, z_0) = (0, 0)$ and $y \equiv \Delta y_0$. The multiple scattering modifies the track parameter of the particle as

$$\frac{d\mathbf{a}}{dy} = T\cdot\mathbf{a} + \begin{pmatrix} 0 \\ \frac{d\Delta a}{dy} \end{pmatrix}, \tag{3.3.50}$$

where

$$\Delta a(y) \equiv a(y) - a(y_0) \tag{3.3.51}$$

represents the change of the dip angle due to the multiple scattering as a function of $y$. This results in

$$\left\{ \begin{array}{lclclcl} \zeta(y) & = & \zeta_0 & + & a_0 y & + & \int^y \Delta a(y) dy \\ a(y) & = & a_0 & + & \Delta a(y) & & \end{array} \right. , \tag{3.3.52}$$

where the parmeters with the suffix 0 are those at the initial point before multiple scattering.

Now, let us assume that we are given $(n+1)$ hit points, $\mathbf{z} \equiv (z_0, \cdots, z_n)^T$, measured at fixed $y$ positions, $y_0 = 0, \cdots, y_n = L$. Then the minimization of the $\chi^2$:

$$\chi^2 = \sum_{i=0}^{n} \left( \frac{z_i - z(y_i, \mathbf{a})}{\sigma_{z_i}} \right)^2 . \tag{3.3.53}$$

The parameter vector that gives the $\chi^2$ minimum is

$$\mathbf{a} \;=\; E_M \cdot \sum_{i=0}^{n} \frac{z_i}{\sigma_{z_i}^2} \begin{pmatrix} 1 \\ y_i \end{pmatrix} , \tag{3.3.54}$$

with

$$E_M^{-1} = \begin{pmatrix} \sum \frac{1}{\sigma_{z_i}^2} & \sum \frac{y_i}{\sigma_{z_i}^2} \\ & \sum \frac{y_i^2}{\sigma_{z_i}^2} \end{pmatrix} . \tag{3.3.55}$$

Through the above equation, the change in $\mathbf{z}$ due to the multiple scattering induces a change in $\mathbf{a}$:

$$\begin{aligned} \Delta\mathbf{a} &= E_M \cdot \begin{pmatrix} 1 & \cdots & 1 \\ y_0 & \cdots & y_n \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sigma_{z_0}^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_{z_1}^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\sigma_{z_n}^2} \end{pmatrix} \cdot \begin{pmatrix} \Delta z_0 \\ \vdots \\ \Delta z_n \end{pmatrix} \\ &= E_M \cdot A \cdot E_{\mathbf{z}}^{-1} \cdot \Delta\mathbf{z}. \end{aligned} \tag{3.3.56}$$

From this we obtain the full error matrix including multiple scattering:

$$\left\langle \Delta\mathbf{a} \cdot \Delta\mathbf{a}^T \right\rangle = E_M \cdot A \cdot E_{\mathbf{z}}^{-1} \cdot \left\langle \Delta\mathbf{z} \cdot \Delta\mathbf{z}^T \right\rangle \cdot E_{\mathbf{z}}^{-1} \cdot A^T \cdot E_M . \tag{3.3.57}$$

Again the problem reduces to the evaluation of the correlation matrix in the coordinate measurements:

$$\langle \Delta z_i \Delta z_j \rangle = \delta_{ij} \sigma_{z_i}^2 + B_{ij} \tag{3.3.58}$$

with

$$B_{ij} \;\equiv\; N' \int [d\Delta a] \; \exp\left[ -\frac{1}{2} \int_0^L \frac{1}{K'} \left( \frac{d\Delta a}{dy} \right)^2 dy \right]$$

$$
\begin{aligned}
& \cdot \left( \int_0^{y_i} \Delta a(y) dy \right) \left( \int_0^{y_j} \Delta a(y) dy \right) \\
= & \frac{K'}{2} y_i^2 (y_j - y_i/3),
\end{aligned}
\tag{3.3.59}
$$

where the last expression is valid for $i \leq j$. The full error matrix is then written in the form:

$$
E = E_M + E_{MS}, \tag{3.3.60}
$$

where the first term is from the coordinate measurement errors while the second term from the multiple scattering:

$$
E_{MS} = E_M \cdot A \cdot E_{\mathbf{z}}^{-1} \cdot B \cdot E_{\mathbf{z}}^{-1} \cdot A^T \cdot E_M. \tag{3.3.61}
$$

In the case of equal space sampling with the same position resolution, $E_M$ and $E_{MS}$ reduce to

$$
\begin{aligned}
E_M & \simeq \frac{2\sigma_z^2}{n} \begin{pmatrix} 2 & -\frac{3}{L} \\ & \frac{6}{L^2} \end{pmatrix} \\
E_{MS} & \simeq \frac{K' L^3}{5} \begin{pmatrix} \frac{1}{21} & \frac{-11}{42L} \\ & \frac{13}{7L^2} \end{pmatrix}
\end{aligned}
\tag{3.3.62}
$$

with

$$
K' = (1 + a^2) \left( \frac{C}{P_T \beta} \right)^2 \frac{(1 + a^2)^{1/2}}{X_0} \tag{3.3.63}
$$

in the large $n$ limit.

# 3.4 Examples of Applications

We have established a general method to calculate the helix parameter error matrix for a high momentum track, taking into account multiple scattering. In this limit, the error matrix is completely determined by the tracking device (its sampling points, coordinate errors, material thickness, ..., etc.). Once the error matrix is given at some pivot, it is straightforward to move the pivot to anywhere in the detector system and to transform the error matrix (propagation of error) accordingly.

## 3.4.a Momentum Resolution

In the case of no multiple scattering, the momentum resolution for a tracking detector with $(n+1)$ equally spaced sampling points is given by Eq.B.9 through

$$
\sigma_\kappa^M = \sigma_{P_T} / P_T^2. \tag{3.4.64}
$$

We plotted $\sigma_\kappa^M$ in Fig.3.1-a) as a function of lever arm length for different numbers of sampling points. Notice that we set $B = 1$ T and $\sigma_x = 100$ $\mu$m there to ease the conversion to different $B$ or $\sigma_x$ values: $\sigma_\kappa^M$ is inversely proportional to $B$, while it is proportional to $\sigma_x$.

The multiple scattering in the tracking volume adds $\sigma_\kappa^{MS}$ given by Eq.3.2.40 to the above measurement error in quardrature (see Eq.3.2.36). Fig.3.1-b) shows $\sigma_\kappa^{MS}/\kappa$ as a function of the lever arm length for different values of material thickness in the tracking device.

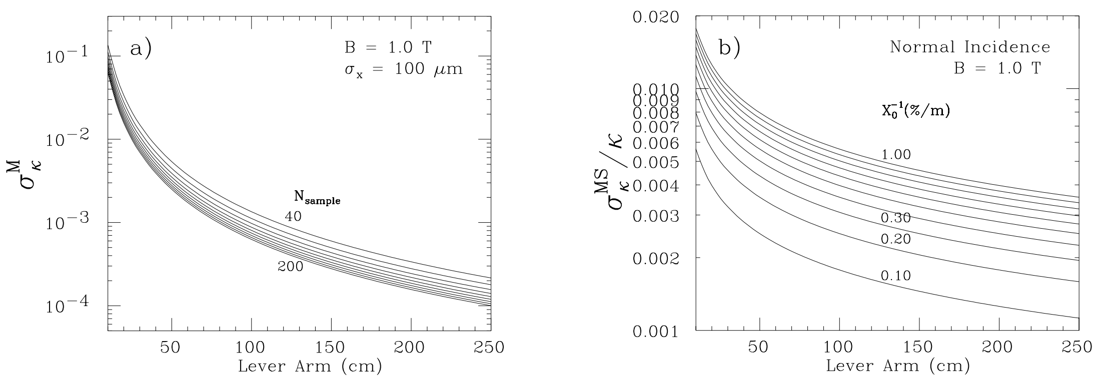


Figure 3.1: (a) The reciprocal transverse momentum resolution as a function of lever arm length for $B = 1$ T, $\sigma_x = 100$ $\mu$m, and $N_{sample} \equiv (n+1) = 40, 60, 80, \cdots, 200$. (b) The multiple scattering contribution as a function of lever arm length for $B = 1$ T and $X_0^{-1} = 0.1, 0.2, \cdots, 1.0\%/\mathrm{m}$.

## 3.4.b Impact Parameter Resolution

Impact parameter resolution $\sigma_\delta$ is calculated as

$$\sigma_\delta^2 = \left(E'^{r\phi}\right)_{11}, \tag{3.4.65}$$

where $E'^{r\phi}$ is the full error matrix obtained from $E^{r\phi}$ given by Eq.3.2.36 by moving the pivot to the interaction point. In the pivot moving we can include the effect of thin-layer multiple scattering, for instance, at the beam pipe wall, according to the method explained in Section 2.4. If the material thickness is non-negligible, we need to use the method in Subsection 3.2.b instead[11]. The effect of external coordinate information can also be implemented as explained there.

Let us consider again a tracking device having $(n+1)$ equally sapced sampling points with an idential spatial resolution $\sigma_x$. For simplicity, we further assume that multiple scattering in the tracking volume is negligible. Then the error matrix is given by Eq.B.7, where the pivot is located at $y_0 = 0$ which is the 0-th hit position. When the 0-th hit is at $r = R_{in}$, we need to transform the error matrix by $\Delta y = -R_{in}$, using Eq.3.2.12. Ignoring the material between the 0-th hit and the interaction point (by setting $E_S = 0$),

[11]This method is applicable also to thin-layer multiple scattering.

we obtain Fig.3.2 which plots $\sigma_\delta/\sigma_x$ as a function of the ratio of the extrapolation length and the lever arm length: $R_{in}/(R_{out} - R_{in})$.

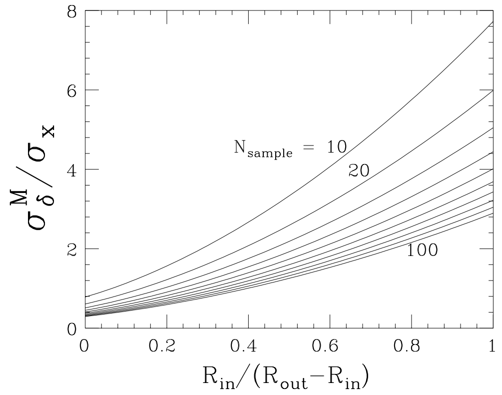


Figure 3.2: The impact parameter resolution normalized by the coordinate error plotted as a function of the ratio of the distance from the innermost hit to the interaction point to the lever arm length for $N_{sample} = 10, 20, \cdots, 100$. No multiple scattering and no external curvature information are assumed.

Notice that the impact parameter resolution here is much worse in general than that expected from a straight-line approximation. This is because of the curvature error. If we have an external tracking device which provides additional curvature information, we can thus improve the impact parameter resolution significantly. We can do this by modifiying $E^{r\phi}$, according to

$$\left(E^{r\phi}\right)^{-1} \to \left(E^{r\phi}\right)^{-1} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\left(\sigma_\kappa^{ext}\right)^2 \end{pmatrix}, \tag{3.4.66}$$

and by following the same steps as above.

## 3.4.c Simple Monte Carlo Simulation of Track Fitting

The full error matrix given by Eqs.3.2.36 and 3.3.60 allows us to smear the exact track parameter given by an event generator, properly taking into account the effects of multiple scattering including the additional correlations among the track parameters. In fact, the $\chi^2$ for the track fitting can be written in the form:

$$\begin{aligned} \chi^2 &= \Delta\mathbf{a}^T \cdot E^{-1} \cdot \Delta\mathbf{a} \\ &= \Delta\mathbf{b}^T \cdot \left(\frac{\partial\mathbf{a}}{\partial\mathbf{b}}\right)^T \cdot E^{-1} \cdot \left(\frac{\partial\mathbf{a}}{\partial\mathbf{b}}\right) \cdot \Delta\mathbf{b}, \end{aligned} \tag{3.4.67}$$

where in the second line we have diagonalized the error matrix:

$$\left(\frac{\partial \mathbf{a}}{\partial \mathbf{b}}\right)^T \cdot E^{-1} \cdot \left(\frac{\partial \mathbf{a}}{\partial \mathbf{b}}\right) = \begin{pmatrix} 1/\sigma_{b_1}^2 & 0 & \cdots & 0 \\ 0 & 1/\sigma_{b_2}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1/\sigma_{b_5}^2 \end{pmatrix}. \tag{3.4.68}$$

Since there is now no correlation among $b_i$'s, we can Gaussian-smear them independently:

$$\Delta b_i = \sigma_{b_i} \cdot \text{(a Gaussian random number centered at zero with unit width)}. \tag{3.4.69}$$

Then the fluctuation of $\mathbf{a}$ is given by

$$\Delta \mathbf{a} = \left(\frac{\partial \mathbf{a}}{\partial \mathbf{b}}\right) \cdot \Delta \mathbf{b}, \tag{3.4.70}$$

where the correlations among the parameters are implemented properly.

# Chapter 4

# Summary

We have developed a general method to handle helical tracks in a uniform magnetic field. The method allows us to propagate track parameters and their error matrix through materials, if any, to anywhere in the detector system in a simple and systematic way. We have then demonstrated that the pivot transformation technique greatly facilitates track manipulations such as track extrapolation, track linking, combined track fitting, vertex fitting, etc. We have described the procedures to achieve these tasks in detail and have summarized the necessary formulae. These formulae are genaral and applicable to any detector configuration, as long as the magnetic field is region-wise uniform so that the track follows a helical trajectory in each region.

We have also examined the high momentum limit, which simplifies the procedure significantly and provides a powerful analytic tool to estimate the performance of a system of tracking devices with an arbitrary configuration: we can estimate the effects of extra coordinate measurements or vertex constraints or combined track fitting on momentum resolution and impact parameter resolution, etc. In the course of this study, we have rederived some well known results in the light of our general method. As an application, we have proposed a simple Monte Carlo method to smear track parameters in accord with the spatial resolution and configuration of a given tracking system, taking into account the error correlations among the parameters and multiple scattering at detector walls or that in the detector sense volumes or both.

# Appendix A

# Effects of Thin Layer Multiple Scattering

Consider a particle passing through a thin layer, whose track parameter vector changes from $\mathbf{a}$ to $\mathbf{a}'$ due to multiple scattering there. We assume here that the track parameter vector of the particle is measured after the multiple scattering to be $\mathbf{a}_M$ with its error matrix $E_{\mathbf{a}_M}$[1]. In the Gaussian approximation valid for small angle multiple scattering, the probability of getting $\mathbf{a}_M$ when the true track parameter vector before the multiple scattering is $\mathbf{a}$ is given by

$$\begin{aligned} P\left(\mathbf{a}_M;\mathbf{a}\right) &= N\int_{-\infty}^{+\infty} d\mathbf{a}' \exp\left[-\frac{1}{2}\left(\left(\mathbf{a}'-\mathbf{a}_M\right)^T \cdot E_{\mathbf{a}_M}^{-1} \cdot \left(\mathbf{a}'-\mathbf{a}_M\right)\right.\right. \\ &\qquad \left.\left. + \left(\mathbf{a}-\mathbf{a}''\right)^T \cdot E'^{-1}_{MS} \cdot \left(\mathbf{a}-\mathbf{a}''\right)\right)\right], \qquad \text{(A.1)} \end{aligned}$$

where $E'_{MS}$ is the diagonalized error matrix corresponding to multiple scattering[2] $N$ is a normalization factor depending only on $E_{\mathbf{a}_M}$ and $E'_{MS}$, and

$$\left(\mathbf{a}-\mathbf{a}'\right) = \left(1+B\right)\left(\mathbf{a}-\mathbf{a}''\right)$$

with $B$ being a matrix whose only nonzero component is $(B)_{35} = \kappa \tan\lambda/(1+\tan^2\lambda)$. Making use of the fact that $B$ is nilpotent, i.e. $B^2 = 0$, we can easily eliminate $\mathbf{a}''$ from the exponent of Eq.A.1:

$$-\frac{1}{2}\left(\left(\mathbf{a}'-\mathbf{a}_M\right)^T \cdot E_{\mathbf{a}_M}^{-1} \cdot \left(\mathbf{a}'-\mathbf{a}_M\right) + \left(\mathbf{a}-\mathbf{a}'\right)^T \left(1-B\right)^T \cdot E'^{-1}_{MS} \cdot \left(1-B\right)\left(\mathbf{a}-\mathbf{a}'\right)\right)$$

[1] As a matter of fact, whether the particle track is measured before or after the multiple scattering does not make any difference in the result.

[2] The multiple scattering only changes $\phi_0$, $\tan\lambda$, and, through the change in $\tan\lambda$, $\kappa$:

$$\Delta\kappa = \kappa\frac{\tan\lambda}{1+\tan^2\lambda}\Delta\tan\lambda. \qquad \text{(A.2)}$$

Notice that the independent variables here are thus $\Delta\phi_0$ and $\Delta\tan\lambda$, $E'_{MS}$ in the above expression is diagonal when we use $\mathbf{a}''$ which is $\mathbf{a}'$ with its $\kappa$ component set equal to that of $\mathbf{a}$. This implies that the components of $E'^{-1}_{MS}$ corresponding to all but $\phi_0$ and $\tan\lambda$ are infinity or in other words those of $E'_{MS}$ are zero. Although $E'^{-1}_{MS}$ is ill-defined in this sense, we can treat the infinite components as if they are finite no matter how large and at the end of calculations let them go to infinity.

$$= -\frac{1}{2}\left(\mathbf{a}''^T \cdot \left(E_{\mathbf{a}_M}^{-1} + E_{MS}^{-1}\right) \cdot \mathbf{a}'' + (\mathbf{a} - \mathbf{a}_M)^T \cdot (E_{\mathbf{a}_M} + E_{MS})^{-1} \cdot (\mathbf{a} - \mathbf{a}_M)\right), \tag{A.3}$$

where we have defined

$$\mathbf{a}'' \equiv \mathbf{a}' - \left(E_{\mathbf{a}_M}^{-1} + E_{MS}^{-1}\right)^{-1} \cdot \left(E_{\mathbf{a}_M}^{-1} \cdot \mathbf{a}_M + E_{MS}^{-1} \cdot \mathbf{a}\right) \tag{A.4}$$

and

$$\begin{aligned} E_{MS}^{-1} &\equiv (1-B)^T E'^{-1}_{MS}(1-B) \\ &= \left[(1+B)E'_{MS}(1+B)^T\right]^{-1}. \end{aligned} \tag{A.5}$$

Changing the integration variable from $\mathbf{a}'$ to $\mathbf{a}''$, Eq.A.1 now becomes

$$\begin{aligned} P(\mathbf{a}_M; \mathbf{a}) &= N \int_{-\infty}^{+\infty} d\mathbf{a}'' \exp\left[-\frac{1}{2}\left(\mathbf{a}''^T \cdot \left(E_{\mathbf{a}_M}^{-1} + E_{MS}^{-1}\right) \cdot \mathbf{a}'' + (\mathbf{a} - \mathbf{a}_M)^T \cdot (E_{\mathbf{a}_M} + E_{MS})^{-1} \cdot (\mathbf{a} - \mathbf{a}_M)\right)\right] \\ &= \exp\left[-\frac{1}{2}(\mathbf{a} - \mathbf{a}_M)^T \cdot (E_{\mathbf{a}_M} + E_{MS})^{-1} \cdot (\mathbf{a} - \mathbf{a}_M)\right] \\ &\quad \cdot N \int_{-\infty}^{+\infty} d\mathbf{a}'' \exp\left[-\frac{1}{2}\mathbf{a}''^T \cdot \left(E_{\mathbf{a}_M}^{-1} + E_{MS}^{-1}\right) \cdot \mathbf{a}''\right]. \end{aligned} \tag{A.6}$$

Notice that the last line is a constant independent of $\mathbf{a}$ and $\mathbf{a}_M$, which proves that the error matrix including the multiple scattering has to be

$$\begin{aligned} E_{\mathbf{a}} &= E_{\mathbf{a}_M} + E_{MS} \\ &= E_{\mathbf{a}_M} + (1+B)E'_{MS}(1+B)^T \end{aligned} \tag{A.7}$$

The derivation of the explicit form of $E'_{MS}$ is straightforward. The multiple scattering changes the helix parameter vector from $\mathbf{a}$ to $\mathbf{a}'$ and, when the pivot is chosen to be the point of the multiple scattering, the tangential vector thereat changes as

$$\begin{aligned} \left.\frac{d}{d\phi}\right|_{\phi=0} &= -\sin\phi_0 \frac{\partial}{\partial x} + \cos\phi_0 \frac{\partial}{\partial y} + \tan\lambda \frac{\partial}{\partial z} \\ &\downarrow \\ \left.\frac{d}{d\phi'}\right|_{\phi'=0} &= -\sin\phi_0' \frac{\partial}{\partial x} + \cos\phi_0' \frac{\partial}{\partial y} + \tan\lambda' \frac{\partial}{\partial z}. \end{aligned} \tag{A.8}$$

Then the direction change of the track in space $\Delta\theta$ is given by

$$\begin{aligned} (\Delta\theta)^2 \simeq \sin^2\Delta\theta &= \left|\frac{\frac{d}{d\phi} \wedge \frac{d}{d\phi'}}{\left|\frac{d}{d\phi}\right| \left|\frac{d}{d\phi'}\right|}\right|^2 \\ &= \frac{(\Delta\tan\lambda)^2 + (1+\tan^2\lambda)(\Delta\phi_0)^2}{(1+\tan^2\lambda)^2} \\ &= \frac{(\Delta\tan\lambda)^2}{(1+\tan^2\lambda)^2} + \frac{(\Delta\phi_0)^2}{(1+\tan^2\lambda)}, \end{aligned} \tag{A.9}$$

where $\Delta\phi_0$ and $\Delta\tan\lambda$ are defined as components of $\Delta\mathbf{a} \equiv \mathbf{a}' - \mathbf{a}$. The above equation determines $E'_{MS}$ through

$$\begin{aligned} \chi^2_{MS} &= \left(\frac{\Delta\theta}{\sigma_{MS}}\right)^2 \\ &= \Delta\mathbf{a}^T \cdot E'^{-1}_{MS} \cdot \Delta\mathbf{a}. \end{aligned} \quad \text{(A.10)}$$

# Appendix B

# Measurement Error Matrix at High Momentum

As shown in Section 3, in the high momentum limit, the $r$-$\phi$ and the $r$-$z$ track fittings are decoupled so that the error matrix or its inverse corresponding to the coordinate measurement errors has a blockwise diagonal form:

$$E_M^{-1} \equiv \frac{1}{2}\left(\frac{\partial^2\chi^2}{\partial\mathbf{a}^T\partial\mathbf{a}}\right) \simeq \begin{pmatrix} \left(E_M^{r\phi}\right)^{-1} & 0 \\ 0 & (E_M^{rz})^{-1} \end{pmatrix}, \tag{B.1}$$

where the component matrices are given by

$$\left(E_M^{r\phi}\right)^{-1} = \begin{pmatrix} \sum \frac{1}{\sigma_{x_i}^2} & -\sum \frac{y_i}{\sigma_{x_i}^2} & \left(\frac{1}{2\alpha}\right) \sum \frac{y_i^2}{\sigma_{x_i}^2} \\ & \sum \frac{y_i^2}{\sigma_{x_i}^2} & -\left(\frac{1}{2\alpha}\right) \sum \frac{y_i^3}{\sigma_{x_i}^2} \\ & & \left(\frac{1}{2\alpha}\right)^2 \sum \frac{y_i^4}{\sigma_{x_i}^2} \end{pmatrix} \tag{B.2}$$

and

$$(E_M^{rz})^{-1} = \begin{pmatrix} \sum \frac{1}{\sigma_{z_i}^2} & \sum \frac{y_i}{\sigma_{z_i}^2} \\ & \sum \frac{y_i^2}{\sigma_{z_i}^2} \end{pmatrix}. \tag{B.3}$$

Notice that, in this limit, the error matrix is determined completely by the $y$-locations[1] of the sampling points and the spatial resolutions thereat.

If $(n+1)$ sampling points are equally spaced and have common resolutions, $\sigma_x$ and $\sigma_z$, then the above equations become

$$\left(E_M^{r\phi}\right)^{-1} = \frac{1}{\sigma_x^2} \begin{pmatrix} n+1 & -L\frac{n+1}{2} & \frac{L^2}{2\alpha}\frac{(n+1)(2n+1)}{6n} \\ & L^2\frac{(n+1)(2n+1)}{6n} & -\frac{L^3}{2\alpha}\frac{(n+1)^2}{4n} \\ & & \left(\frac{L^2}{2\alpha}\right)^2 \frac{(n+1)(2n+1)(3n^2+3n-1)}{30n^3} \end{pmatrix} \tag{B.4}$$

[1]In our coordinate system, $y$ corresponds to $r$.

and

$$(E_M^{rz})^{-1} = \frac{1}{\sigma_z^2} \begin{pmatrix} n+1 & L\frac{n+1}{2} \\ & L^2\frac{(n+1)(2n+1)}{6n} \end{pmatrix}, \tag{B.5}$$

where we have defined the lever arm length $L$ by

$$L \equiv r_{out} - r_{in}. \tag{B.6}$$

Notice that in the above equations, we have shown only upper triangles of the matrices, since they are all symmetric. By matrix inversions, we thus obtain

$$E_M^{r\phi} = \sigma_x^2 \begin{pmatrix} \frac{3(3n^2+3n+2)}{(n+1)(n+2)(n+3)} & \frac{18n(2n+1)}{L(n+1)(n+2)(n+3)} & \frac{60\alpha n^2}{L^2(n+1)(n+2)(n+3)} \\ & \frac{12n(2n+1)(8n-3)}{L^2(n-1)(n+1)(n+2)(n+3)} & \frac{360\alpha n^3}{L^3(n-1)(n+1)(n+2)(n+3)} \\ & & \frac{720\alpha^2 n^3}{L^4(n-1)(n+1)(n+2)(n+3)} \end{pmatrix} \tag{B.7}$$

and

$$E_M^{rz} = \sigma_z^2 \begin{pmatrix} \frac{2(2n+1)}{(n+1)(n+2)} & -\frac{6n}{L(n+1)(n+2)} \\ & \frac{12n}{L^2(n+1)(n+2)} \end{pmatrix}. \tag{B.8}$$

The above error matrices are defined with the pivotal point $(x_0, y_0, z_0)$ chosen to be at the 0-th hit.

From the above formulae, we can estimate, for instance, the transverse momentum resolution or $\sigma_\kappa$ as

$$\begin{aligned} \sigma_\kappa^M &= \left(\frac{\alpha' \sigma_x}{L^2 B}\right) \sqrt{\frac{720n^3}{(n-1)(n+1)(n+2)(n+3)}} \\ &\simeq \left(\frac{\alpha' \sigma_x}{L^2 B}\right) \sqrt{\frac{720}{n+5}}, \end{aligned} \tag{B.9}$$

where the last line is none other thatn the familiar text book expression for the momentum resolution valid in the large $n$ limit.

# Appendix C

# Multiple Scattering Error Matrix at High Momentum

In the high momentum limit, the error matrix takes a blockwise diagonal form.

$$E \simeq \begin{pmatrix} E^{r\phi} & 0 \\ 0 & E^{rz} \end{pmatrix}, \tag{C.1}$$

where each of the component matrices has two subcomponents corresponding to measurement and multiple scattering errors. The measurement error matrices are already treated in Appendix B. The remaining task is to calculate their multiple scattering counter parts.

In the following, we assume that there are $(n+1)$ sampling points equally spaced in a uniform tracking medium. Under this assumption, we can derive the following results from Eqs.3.2.35 to 3.2.37:

$$\begin{aligned}
\left(E^{r\phi}_{MS}\right)_{11} &= K \cdot \frac{L^3(n-2)(n-1)(2n^3+489n^2+481n-1299)}{1260n^2(n+1)(n+2)(n+3)} \\
\left(E^{r\phi}_{MS}\right)_{12} &= K \cdot \frac{L^2(n-2)(14n^4+2631n^3+4526n^2-3025n+834)}{840n^2(n+1)(n+2)(n+3)} \\
\left(E^{r\phi}_{MS}\right)_{13} &= K \cdot \frac{(2\alpha)L(n-2)(n^3+600n^2+1121n-726)}{168n(n+1)(n+2)(n+3)} \\
\left(E^{r\phi}_{MS}\right)_{22} &= K \cdot \frac{L(16n^6+1862n^5+1299n^4-6182n^3+2065n^2-1886n-24)}{70(n-1)n^2(n+1)(n+2)(n+3)} \\
\left(E^{r\phi}_{MS}\right)_{23} &= K \cdot \frac{(2\alpha)(3n^5+869n^4+787n^3-2969n^2+650n-480)}{28(n-1)n((n+1)(n+2)(n+3)} \\
\left(E^{r\phi}_{MS}\right)_{33} &= K \cdot \frac{5(2\alpha)^2n(n^3+112n^2+135n-362)}{14L(n-1)(n+1)(n+2)(n+3)}
\end{aligned} \tag{C.2}$$

and

$$E^{rz}_{MS} = K \cdot \begin{pmatrix} \frac{L^3(n-1)(6n^3+177n^2+163n-172)}{630n^2(n+1)(n+2)} & -\frac{L^2(n-1)(22n^3+390n^2+523n-122)}{420n^2(n+1)(n+2)} \\ & \frac{L(26n^4+265n^3+287n^2-166n+8)}{70n^2(n+1)(2n+1)} \end{pmatrix}. \tag{C.3}$$

The above error matrices are defined with the pivotal point $(x_0, y_0, z_0)$ chosen to be at the 0-th hit. Notice also that we have shown only upper triangles of the matrices, since they are all symmetric.