Basic Physics Behind Operation of TPC

Part II

-- Application to MPGD Readout TPCs --

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Coordinate Measurements
Charge Centroid Method with Readout Pads

Fundamental Limits on Spatial Resolution for a MPGD Readout TPC

Analytic Formulation of Spatial Resolution

Important Outcome from KEK Beam Tests (Asia, Europe, and North America)
MPGD Readout TPC

Coordinate Measurement Process

Coordinate System
We set our coordinate system in such a way that the readout pads are arranged in a row to measure the x-coordinate with charge centroid method, the y-coordinate from the pad row number, and the z-coordinate from the drift time.

Basic Assumptions
For simplicity, we will consider for a while a charged particle at normal incidence. We also assume that the effect of delta-rays is negligible (good approximation if there is a strong enough B-field) so that all the track electrons can be regarded as starting from a single point when projected to the (x, z) plane. These track electrons drift towards the amplification region while experiencing diffusion. The track electrons are then gas amplified while experiencing further diffusion. As we have discussed, when we readout pad signals with a slow enough electronics, only the real charge arriving at a readout pad counts. The spatial width of the signal is then determined by the width of the real charge distribution on the pad plane as determined by the diffusion in the drift and the amplification regions. Notice that we are dealing with normal incidence for which angular pad effect is absent.

In what follows we start from an ideal situation with a perfect readout plane, switching on one-by-one complications expected for more realistic situations.
**Fundamental Processes**

- **Amplification Gap**
- **Drift Volume**
- **Readout Pads**

**Beam**

**Ionizations**

$P_I(N; \bar{N})$  
Normal incidence  
(no angle effect)  
No $\delta$-ray

**Drift and Diffusion**

$P_D(x_i; \sigma_d) = \frac{1}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{x_i^2}{2\sigma_d^2}\right)$

$\sigma_d = C_d\sqrt{z}$

**Amplification and further Diffusion**

$P_G(G/\bar{G}; \theta) = \frac{(\theta + 1)^{\theta+1}}{\Gamma(\theta + 1)} \left(\frac{G}{\bar{G}}\right)^\theta \exp\left(- (\theta + 1) \left(\frac{G}{\bar{G}}\right)\right)$

**Pad Response**

**Coordinate**
**Ionization Statistics**

**Ideal Readout Plane: Coordinate = Simple C.O.G.**

**PDF for C.O.G. of N electrons**

We assume here an ideal readout plane that can measure the $x$-coordinates of individual track electrons exactly. The probability distribution function for the center of gravity of $N$ track electrons is given by

$$P(\bar{x}) = \sum_{N=1}^{\infty} P_I(N; N) \prod_{i=1}^{N} \left( \int dx_i P_D(x_i; \sigma_d) \right) \delta \left( \bar{x} - \frac{1}{N} \sum_{i=1}^{N} x_i \right)$$

where $P_I(N; N)$ is the ionization statistics and $P_D(x_i; \sigma_d)$ is the Gaussian diffusion function.

$$P_D(x_i; \sigma_d) = \frac{1}{\sqrt{2\pi}\sigma_d} \exp \left( - \frac{x_i^2}{2\sigma_d^2} \right)$$

$$\sigma_d = C_d \sqrt{z}$$

where $C_d$ is the diffusion coefficient and $z$ is the drift length. The track is assumed to passed through the TPC at $x=0$ parallel with the readout plane and perpendicular to the pad rows.

The center of gravity of the $N$ electrons is the best possible estimator of the incident $x$-coordinate of the track:

$$\langle \bar{x} \rangle := \int d\bar{x} P(\bar{x}) \bar{x} = 0$$

The variance of the C.O.G. is then given by

$$\sigma_{\bar{x}}^2 := \int d\bar{x} P(\bar{x}) \bar{x}^2 = \sigma_d^2 \left( \frac{1}{N} \right) =: \sigma_d^2 \frac{1}{N_{\text{eff}}}$$

by definition. This leads us to

$$N_{\text{eff}} := \frac{1}{\langle 1/N \rangle} < \langle N \rangle$$

What decides the spatial resolution is not the average number of ionization electrons but the inverse of the average of its inverse.
Gas Gain Fluctuation

Coordinate = Gain-Weighted Mean

PDF for gain-weighted mean

We now switch on the gas gain fluctuation and assume that the coordinate measured by the readout plane is the gain-weighted mean of the N ionization electrons.

\[ P(\bar{x}) = \sum_{N=1}^{\infty} P_I(N; \bar{N}) \prod_{i=1}^{N} \left[ \int dx_i P_D(x_i; \sigma_d) \right. \]

\[ \times \int d \left( \frac{G_i}{\bar{G}} \right) P_D \left( \frac{G_i}{\bar{G}}; \theta_{pol} \right) \right] \delta \left( \bar{x} - \sum_{i=1}^{N} \frac{G_i x_i}{\sum_{i=1}^{N} G_i} \right) \]

Gas gain fluctuation

Gain-weighted mean

We used the Polya parameter as an index even though the PG is non-Polya in general.

Notice that

\[ \sum_{i=1}^{N} G_i \approx N \bar{G} \]

Again we assume that the charged particle passed through the TPC at \( x=0 \) parallel with the readout plane and perpendicular to the pad rows.

The average of the gain-weighted mean has then no bias

\[ \langle \bar{x} \rangle := \int d\bar{x} \ P(\bar{x}) \ \bar{x} = 0 \]

The variance of the C.O.G. is then given by

\[ \sigma_{\bar{x}}^2 := \int d\bar{x} \ P(\bar{x}) \ \bar{x}^2 \approx \sigma_d^2 \left( \frac{1}{N} \right) \left( \frac{\bar{G}}{G} \right)^2 =: \sigma_d^2 \frac{1}{N_{\text{eff}}} \]

where use has been made of

\[ \sum_{i=1}^{N} G_i \approx N \bar{G} \]

We hence have

\[ N_{\text{eff}} := \left[ \langle \frac{1}{N} \rangle \left( \frac{\bar{G}}{G} \right)^2 \right]^{-1} = \frac{1}{\langle 1/N \rangle} \left( \frac{1 + \theta_{pol}}{2 + \theta_{pol}} \right) < \langle N \rangle \]

The gas gain fluctuation therefore further reduces the effective number of electrons.
In the case of Snyder’s model, gain fluctuation is exponential and \( K = 1 \) (theta=0) and the Neff is reduced by a factor of 2 by it. In the case of Legler’s model, theta>0 and the reduction is less severe. If we assume theta=0.5, for instance, we have a factor of 1.5 reduction:

\[
N_{\text{eff}} = \left[ \langle \frac{1}{N} \rangle \langle \left( \frac{G}{G} \right)^2 \rangle \right]^{-1} = 21 < \langle N \rangle = 71
\]
Finite Size Pads

Coordinate = Charge Centroid

PDF for charge centroid

We now replace the continuous readout plane with an array of finite size pads. The finite size pads break the translational symmetry. We hence need to specify the track position relative to the pad center. The arrival point of i-th ionization electron is given by

\[ x_i = \bar{x} + \Delta x_i \]

The charge centroid is then given by

\[ \bar{x} = \frac{\sum_a Q_a (a w)}{\sum_a Q_a} \]

with \( w \) being the pad pitch. The probability distribution function for charge centroid is

\[
P(\bar{x}; \tilde{x}) = \prod_{i=1}^{\infty} P_i(N; \tilde{N}) \prod_{i=1}^{N} \left[ \int d\Delta x_i \, P_D(\Delta x_i; \sigma_d) \int d \left( \frac{G_i}{G} \right) \, P_G \left( \frac{G_i}{G}; \theta_{pad} \right) \right] \times \prod_{a} \left[ \int d\Delta Q_a \, P_E(\Delta Q_a; \sigma_E) \int dQ_a \, \delta \left( Q_a - \sum_{i=1}^{N} G_i \, F_a(\bar{x} + \Delta x_i) - \Delta Q_a \right) \right] \times \delta \left( x - \frac{\sum_a Q_a (a w)}{\sum_a Q_a} \right)
\]

where \( F_a \) is the normalized pad response function for a-th pad

\[ \sum_{a} F_a(\bar{x} + \Delta x_i) = 1 \]

\( G_i \) is the gas gain for the i-th ionization electron, and \( \Delta Q_a \) is the electronic noise:

\[ \langle (\Delta Q_a)^2 \rangle = \sigma_E^2 \]
In order to take into account the effect of finite size pads as known as the S-shape systematics, we define the variance by

\[ \sigma_x^2 := \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\left(\frac{\tilde{x}}{w}\right) \int d\bar{x} P(\bar{x}; \tilde{x}) (\bar{x} - \tilde{x})^2 \]

Substituting the PDF given above in this and with some arithmetics, we obtain

\[ \sigma_x^2 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\left(\frac{\tilde{x}}{w}\right) \left[ [A] + \frac{1}{N_{\text{eff}}} [B] \right] + [C] \]

where

\[ [A] := \left( \sum_a (a w) \langle F_a(\tilde{x} + \Delta x) \rangle - \tilde{x} \right)^2 \]

is a purely geometric term corresponding to the S-shape systematics due to the finite pad pitch and disappears rapidly as \(z\) increases. On the other hand,

\[ [B] := \sum_{a,b} a b w^2 \langle F_a(\tilde{x} + \Delta x) F_b(\tilde{x} + \Delta x) \rangle \]

\[ - \left( \sum_a a w \langle F_a(\tilde{x} + \Delta x) \rangle \right)^2 \]

is a term representing the contributions from diffusion, gas gain fluctuation, and finite pad pitch. The contribution of this term scales as \(1/N_{\text{eff}}\) and dominates the spatial resolution at a long drift distance. The last term

\[ [C] := \left( \frac{\sigma_E}{G} \right)^2 \left\langle \frac{1}{N^2} \right\rangle \sum_a (a w)^2 \]

is an electronic noise term, which is \(z\)-independent and scales as \(\langle 1/N^2 \rangle\).

The correlation function and the averaged pad response functions are defined by

\[ \langle F_a(\tilde{x} + \Delta x) F_b(\tilde{x} + \Delta x) \rangle := \int d\Delta x P_D(\Delta x; \sigma_d) F_a(\tilde{x} + \Delta x) F_b(\tilde{x} + \Delta x) \]

and

\[ \langle F_a(\tilde{x} + \Delta x) \rangle := \int d\Delta x P_D(\Delta x; \sigma_d) F_a(\tilde{x} + \Delta x) \]

An asymptotic form at large \(z\) reads

\[ \sigma_x^2 \simeq \sigma_0^2 + \frac{1}{N_{\text{eff}}} C_d^2 z \]

with

\[ \sigma_0^2 := \frac{1}{N_{\text{eff}}} \left[ \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\left(\frac{\tilde{x}}{w}\right) [A](z = 0) \right] + [C] \]
Interpretation

[A] Purely geometric term (S-shape systematics from finite pad pitch): rapidly disappears as $Z$ increases

[B] Diffusion, gas gain fluctuation & finite pad pitch term: scales as $1/N_{eff}$

$$\sigma^2_x \simeq \frac{1}{N_{eff}} \left[ \int_{-\frac{1}{2}}^{+\frac{1}{2}} d \left( \frac{x}{w} \right) [A](z = 0) + C_d^2 z \right]$$

For delta-fun like PRF asymptotically:

$$\sigma^2_x \simeq \frac{1}{N_{eff}} \left( \frac{w^2}{12} + C_d^2 z \right)$$

[C] Electronic noise term: $Z$-independent, scales as $\langle 1/N^2 \rangle$
Application to Micromegas

For a delta-function like PRF, there is a scaling law: $\sigma_x/w$ depends only on $\sigma_d/w$ and $N_{eff}$.

The formula has a fixed point

$(0, 1/\sqrt{12})$

Full formula enters asymptotic region at

$\sigma_d/w \simeq 0.4$

Full formula has a minimum of

$\sigma_x/w \simeq 0.1$

$\sigma_d/w \simeq 0.3$

(0, 1/\sqrt{12}) : hodoscope limit

$N_{eff}=18.5$

Full Theory

Asymptotic Formula

No Noise

$\sigma_0/w = 1/\sqrt{12 \, N_{eff}}$

$\sigma_0/w = 1/\sqrt{12 \, N_{eff}}$
Comparison with MC

- Theory reproduces the Monte Carlo simulation very well!
- We can estimate the resolution analytically

\[ \sigma_x = \sigma_x(z; w, C_d, N_{eff}, [f_j]) \]

- pad pitch
- diffusion const.
- pad response function

\[ \delta \text{-fun. for MM: } \sigma_{PRF} \approx 12\mu m \]
\[ \text{gauss. for GEM: } \sigma_{PRF} \approx 350\mu m \]
Comparison with Measurements

**KEK beam test data**

- Theory reproduces the data well.
- Underestimation in the data of $\sigma_x$ at short drift distance is due to track bias caused by S-shape systematics.
- Global likelihood method eliminates the S-shape systematics at short distance when possible and hence gives better resolution than the simple charge centroid method used in the chi-square fit.

**MP-TPC Micromegas**

ArIso(95:5), B=1T

$E_{\text{drift}} = 220\text{V/cm}$

$w = 2.3\text{mm}$
Extrapolation to LC TPC

Need to reduce pad size relative to PRF

- Resistive anode for MM.
- Digital pixel readout for MM corresponding to an ideal readout plane to avoid the effect of gain fluctuation (the best if feasible).

Defocusing + narrow (1mm) pad for GEM.

Recent results seem promising for both resistive anode and digital pixel readout schemes (Paul’s talk)!
In the case of GEM, there is no simple scaling as with micromegas, since there is an additional dimensionful parameter that is the intrinsic signal width ($\sigma_{PRF}$) which is determined by the diffusion in the transfer and induction gaps. When it is large enough compared with the pad pitch we can avoid the hodoscope effect at a short drift distance.

The TU-TPC data indicates

$$N_{eff} = \frac{128}{22} = 34 \pm 4$$

in good agreement with the MP-TPC result.
The average charge on a-th pad

The average charge on a-th pad is given by

\[ \langle Q_a(\tilde{x}) \rangle = \bar{N} G \langle F_a(\tilde{x} + \Delta x) \rangle \]

resulting in the average charge fraction

\[ \langle Q_a(\tilde{x}) \rangle / (\bar{N} \bar{G}) = \langle F_a(\tilde{x} + \Delta x) \rangle \]

where

\[ \hat{x} := a w - \tilde{x} \]

is the location of the pad center measured from the incident position of the track and

\[ \sigma^2 := \sigma_{\text{PRF}}^2 + \sigma_d^2 = \sigma_{\text{PRF}}^2 + C_d^2 z \]

is the squared sum of the intrinsic width of the pad response function at z=0 and the width due to diffusion in the drift region.

We can hence define a normalized apparent pad response function

\[ Q_{\text{PR}}(\hat{x}) := \frac{1}{w} \int_{-w/2}^{+w/2} \! \! d\xi \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{1}{2} \left( \frac{\hat{x} + \xi}{\sigma} \right)^2 \right] \]

which has the variance

\[ \sigma_{\text{PR}}^2 = \int d\hat{x} Q_{\text{PR}}(\hat{x}) \hat{x}^2 \]

\[ = \frac{1}{w} \int_{-w/2}^{+w/2} \! \! d\xi \int_{-\infty}^{+\infty} \! \! d\hat{x} \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{1}{2} \left( \frac{\hat{x} + \xi}{\sigma} \right)^2 \right] \hat{x}^2 \]

\[ = \frac{1}{w} \int_{-w/2}^{+w/2} \! \! d\xi (\sigma^2 + \xi^2) = \sigma^2 + \frac{w^2}{12} \]

From this we obtain

\[ \sigma_{\text{PR}}^2(0) := \sigma_{\text{PR}}^2 - C_d^2 z = \sigma_{\text{PRF}}^2 + \frac{w^2}{12} \]

By plotting \( \sigma_{\text{PR}}^2 \) as a function of \( z \), we can hence extract \( C_d \) from the slope and \( \sigma_{\text{PRF}}^2 \) from the intercept with the finite pad pitch correction of \( w^2/12 \).
Cd Measurement

TU-TPC test at KEK cryo hall (Dec. 2007)

Assuming

\[ v_{drift} = 4.2 \text{ [cm/\mu s]} \]

from drift time data, plot the apparent pad response function as a function of the drift distance. Then perform a straight-line fit.

From the slope

\[ C_D = 128 \text{ [\mu m/\sqrt{cm}]} \]

From the intercept

\[ \sigma_{PR}^2 (0) = \sigma_{PR}^2 - C_d^2 z = \sigma_{PRF}^2 + \frac{w^2}{12} \]

\[ \sigma_{PRF}^2 = \sigma_{PR}(0)^2 - \frac{w^2}{12} = (270 \text{ [\mu m]})^2 \]

roughly consistent with what we expect from the diffusion in transfer and induction gaps.
Angular Pad Effect
Resolution degradation for inclined tracks

Consider an inclined track having an angle \( \phi \) to the \( yz \) plane and an angle \( \theta \) to the \( xy \) plane (pad plane). The projection of the track electrons to the \( xz \) plane is no longer point-like even if the cluster size is negligible for secondary ionizations. This extra charge spread adds up to that caused by diffusion. Consequently, the statistical fluctuation of the locations of the primary ionizations as well as that of the secondary ionizations cause additional contributions to the coordinate measurement error. The effect is further amplified by the gas gain fluctuations. The degradation of spatial resolution due to the finite \( \phi \) is known as the angular pad effect and is inevitable as long as we use ordinary readout pads, since they break the rotational symmetry in the \( \phi \) direction (notice that the symmetry breaking must be much softer in the case of pixel readout). If the \( \theta \) is nonzero, the drift distance depends on where you are on the track and the average number of ionization electrons will be larger due to the longer track segment per pad row. As long as we use a short enough pad, the drift distance can be regarded as approximately constant within a pad row. We can hence assume that the effect of the finite \( \theta \) can be taken into account by scaling \( N_{eff} \) by the amount expected from the increase of the track segment length. For this reason we assume in what follows that the \( \theta \) is zero unless otherwise stated. We again start from an ideal situation.
Ionization Statistics

Ideal Readout Plane: Coordinate = Simple C.O.G.

PDF for C.O.G.
We assume here an ideal readout plane that can measure the x-coordinates of individual track electrons exactly. The probability distribution function for the center of gravity is given by

$$P(x) = \sum_{N=1}^{\infty} P_{P1}(N) \prod_{i=1}^{N} \left[ \int \frac{ds_i}{l} \sum_{M_i=1}^{\infty} P_{S1}(M_i) \right] \times \prod_{j=1}^{M_i} \left[ \int d\Delta x_{ij} d\Delta y_{ij} P_D(\Delta x_{ij}) P_D(\Delta y_{ij}) \right] \times \delta \left( \bar{x} - \frac{\sum_{i,j} x_{ij} \theta (\frac{L}{2} + y_{ij}) \theta (\frac{L}{2} - y_{ij})}{\sum_{i,j} \theta (\frac{L}{2} + y_{ij}) \theta (\frac{L}{2} - y_{ij})} \right)$$

where $Cd$ is the diffusion coefficient and $z$ is the drift length. The “L” is the projected track length to the xy plane and $s_i$ is the projected location of $i$-th cluster along the track. The arrival point of $j$-th electron in the $i$-th cluster is $(x_{ij}, y_{ij})$:

$$x_{ij} = \bar{x} + s_i \sin \phi + \Delta x_{ij}$$
$$y_{ij} = s_i \cos \phi + \Delta y_{ij}$$

where $s_i=0$ is in the middle of the row in question. The “L” is the height of the row. The average $x$-coordinate of the center of gravity of the track electrons arriving at the row is

$$\langle \bar{x} \rangle := \int d\bar{x} P(\bar{x}) \bar{x} = \bar{x}$$

since $s_i$, delta $x_{ij}$, and delta $y_{ij}$ average to zero.
**Variance of the C.O.G.**

By definition the variance of the C.O.G. is given by

\[
\sigma_{\bar{x}}^2 := \int d\bar{x} \, P(\bar{x}) \, (\bar{x} - \langle \bar{x} \rangle)^2
\]

Substituting

\[
\langle \bar{x} \rangle = \tilde{x}
\]

we have

\[
x_{ij} = \tilde{x} + s_i \sin \phi + \Delta x_{ij}
\]

we have

\[
\sigma_{\bar{x}}^2 = \sum_{N=1}^{\infty} P_{PL}(N) \prod_{i=1}^{N} \left[ \int \frac{ds_i}{\ell} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right] \prod_{j=1}^{M_i} \int d\Delta x_{ij} \, d\Delta y_{ij} \, P_D(\Delta x_{ij}) \, P_D(\Delta y_{ij}) \times \left( \frac{\sum_{i,j} (s_i \sin \phi + \Delta x_{ij}) \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right)}{\sum_{i,j} \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right)} \right)^2
\]

The delta \(x_{ij}\) integral is straightforward

\[
\sigma_{\bar{x}}^2 = \sum_{N=1}^{\infty} P_{PL}(N) \prod_{i=1}^{N} \left[ \int \frac{ds_i}{\ell} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right] \prod_{j=1}^{M_i} \int d\Delta y_{ij} \, P_D(\Delta y_{ij}) \times \left[ \sin^2 \phi \sum_i s_i^2 \left( \sum_j \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right) \right)^2 + \sigma_d^2 \sum_{i,j} \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right) \right] \left( \sum_{i,j} \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right) \right)^2
\]

since cross terms just vanish.

We can cast this into the form:

\[
\sigma_{\bar{x}}^2 = \sin^2 \phi \left\langle \sum_i s_i^2 \left( \sum_j \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right) \right)^2 \right\rangle + \sigma_d^2 \left\langle \sum_{i,j} \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right) \right\rangle
\]

where the 1st term on the R.H.S. is given by

\[
\sigma_{\bar{x};_{\text{ang}}}^2 = \sin^2 \phi \sum_{N=1}^{\infty} P_{PL}(N) \prod_{i=1}^{N} \left[ \int \frac{ds_i}{\ell} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right] \prod_{j=1}^{M_i} \int d\Delta y_{ij} \, P_D(\Delta y_{ij}) \sum_i s_i^2 \left( \sum_j \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right) \right)^2 \left( \sum_{i,j} \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right) \right)^2
\]

while the 2nd term by

\[
\sigma_{\bar{x};_{\text{diff}}}^2 = \sigma_d^2 \sum_{N=1}^{\infty} P_{PL}(N) \prod_{i=1}^{N} \left[ \int \frac{ds_i}{\ell} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right] \prod_{j=1}^{M_i} \int d\Delta y_{ij} \, P_D(\Delta y_{ij}) \right\rangle \frac{1}{\sum_{i,j} \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right)}
\]

Notice that the 2nd term defines \(\text{Neff}\) by

\[
\frac{1}{\text{Neff}} := \sum_{N=1}^{\infty} P_{PL}(N) \prod_{i=1}^{N} \left[ \int \frac{ds_i}{\ell} \sum_{M_i=1}^{\infty} P_{SI}(M_i) \right] \prod_{j=1}^{M_i} \int d\Delta y_{ij} \, P_D(\Delta y_{ij}) \frac{1}{\sum_{i,j} \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right)}
\]

which is a more microscopic definition in
terms of the combined effects of primary and secondary ionization statistics together
with diffusions. It’s worth noting that

\[ N_{\text{acc}} := \sum_{i,j} \theta \left( \frac{L}{2} + y_{ij} \right) \theta \left( \frac{L}{2} - y_{ij} \right) \]

just counts the number of track electrons accepted by the row in question. If the
track is far away from the readout plane, the memory about parent clusters will be
lost by the time individual track electrons arrive at the readout plane because of
diffusions. On the other hand, if the track is near the readout plane, the correlation
among the track electrons belonging to the same parent cluster is very strong and
hence they are either all accepted or all rejected by the row and hence the Neff
value will be smaller than those at longer drift distances. This clustering effect
should be there even in the case of \( \phi = 0 \). Neff is in principle \( z \)-dependent!

The probability of a secondary electron at \( s_i \) reaches the row in question is given by

\[ \eta(s_i \cos \phi) := \int_{-\frac{L}{2} - s_i \cos \phi}^{\frac{L}{2} - s_i \cos \phi} \exp \left( -\frac{(\Delta y)^2}{2\sigma_d^2} \right) \]

With this the probability of \( k \) electrons created at \( s_i \) reach the row is written in
the form:

\[ \bar{P}_{SI}(k; y_i) := \sum_{M=1}^{\infty} P_{SI}(M) \frac{M!}{k!(M - k)!} \eta(y_i)^k (1 - \eta(y_i))^{M-k} \]

Then we can rewrite the P.D.F. for the C.O.G. as

\[ P(\bar{x}) = \sum_{N=1}^{\infty} P_{PI}(N) \prod_{i=1}^{N} \left[ \int \frac{ds_i}{l} \sum_{k_i=0}^{\infty} \bar{P}_{SI}(k_i; s_i \cos \phi) \right] \left[ \prod_{j=1}^{k_i} \int d\Delta x_{ij} P_D(\Delta x_{ij}; \sigma_d) \right] \times \delta \left( \bar{x} - \sum_{i=1}^{N} \sum_{j=1}^{k_i} (\bar{x} + s_i \sin \phi + \Delta x_{ij}) \right) \]

where

\[ P_{PI}(N) := \frac{\tilde{N}^N}{N!} \exp(-\tilde{N}) \]

and

\[ P_D(\Delta x; \sigma_d) := \frac{1}{\sqrt{2\pi\sigma_d}} \exp \left( -\frac{(\Delta x)^2}{2\sigma_d^2} \right) \]
From these we have

\[ \sigma_{\text{diff}}^2 = \sigma_d^2 \sum_{N=1}^{\infty} P_{P1}(N) \sum_{k_1, \ldots, k_N} \prod_{i=1}^{N} \left[ \int \frac{dy_i}{l \cos \phi} P_{SI}(k_i; y_i) \right] \left( \frac{1}{\sum_{i=1}^{N} k_i} \right) \]

which implies

\[ N_{\text{eff}} := \left[ \sum_{N=1}^{\infty} P_{P1}(N) \sum_{k_1, \ldots, k_N} \prod_{i=1}^{N} \left( \int \frac{dy_i}{l \cos \phi} \tilde{P}_{SI}(k_i; y_i) \right) \left( \frac{1}{\sum_{i=1}^{N} k_i} \right) \right]^{-1} \]

Defining

\[ \tilde{P}(k_i) := \int \frac{dy}{l \cos \phi} \tilde{P}_{SI}(k_i; y) \]

We can further reduce this formula for Neff to

\[ N_{\text{eff}} = \left[ \sum_{N=1}^{\infty} P_{P1}(N) \sum_{k_1, \ldots, k_N} \prod_{i=1}^{N} \left( \tilde{P}_{SI}(k_i) \right) \left( \frac{1}{\sum_{i=1}^{N} k_i} \right) \right]^{-1} \]

For a row with a height of L=6.3mm, Neff has been calculated as a function of the relative diffusion, sigma_d/L, assuming Ar 100% and normal incidence and plotted in the following figure. We can see the effect of de-clustering that appears as a slight increase of Neff with the drift distance, being consistent with our naive expectation.

Notice that for an LC-TPC, for which the Cd value is expected to be less than 50 microns/sqrt(cm), sigma_d/L never exceeds 0.2 even for a drift length less than 200 cm.

The Neff value is hence approximately the drift-length independent, justifying our analytic formula for the phi=0 case.
Angular effect

The angular effect is contained in

\[ \sigma_{x:ang}^2 = \tan^2 \phi \sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1, \ldots, k_N} \prod_{i=1}^{N} \left[ \int \frac{dy_i}{l \cos \phi} \tilde{P}(k_i; y_i) \right] \left( \sum_{i=1}^{N} k_i^2 y_i^2 \right)^2 \]

Defining

\[ \langle y^2 \rangle_{k_i} := \int \frac{dy}{l \cos \phi} \tilde{P}(k_i; y) y^2 \]

and recalling

\[ \bar{P}(k_i) := \int \frac{dy}{l \cos \phi} \tilde{P}(k_i; y) \]

we can rewrite this formula as

\[ \sigma_{x:ang}^2 = \tan^2 \phi \sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1, \ldots, k_N} \prod_{i \neq i'} \bar{P}(k_i) \left( \sum_{i=1}^{N} k_i^2 \langle y^2 \rangle_{k_i} \right)^2 \]

\[ = \tan^2 \phi \sum_{N=1}^{\infty} P_{PI}(N) N \sum_{k_1, \ldots, k_N} \prod_{i=2}^{N} \bar{P}(k_i) \left( \sum_{i=1}^{N} k_i^2 \langle y^2 \rangle_{k_1} \right)^2 \]

Notice that

\[ P_{PI}(N) := \frac{\bar{N}^N}{N!} \exp(-\bar{N}) \]

deepends on \( \phi \) through \( \bar{N} \). On the other hand, \( \langle y^2 \rangle_k \) and \( \bar{P}(k) \) are angle-independent since

\[ l \cos \phi = y_{\max} - y_{\min} =: \Delta Y \]

It is probably useful to further rewrite the formula in the following form:

\[ \sigma_{x:ang}^2 = \frac{L^2}{12 \bar{N}_{eff}} \tan^2 \phi \]

where

\[ \bar{N}_{eff} := \left[ \frac{12}{L^2} \sum_{N=1}^{\infty} P_{PI}(N) N \sum_{k_1, \ldots, k_N} \prod_{i=2}^{N} \bar{P}(k_i) \left( \sum_{i=1}^{N} k_i^2 \langle y^2 \rangle_{k_i} \right)^2 \right]^{-1} \]

The following is a sample calculation for \( L = 6.3 \text{ [mm]} \).
Short drift limit

In the short drift limit, the diffusion becomes negligible and its corresponding P.D.F. becomes a delta-function and hence we have

\[ \eta(y) \rightarrow \theta \left( \frac{L}{2} + y \right) \theta \left( \frac{L}{2} - y \right) \quad \text{as} \quad \frac{\sigma_d}{L} \rightarrow 0 \]

leading us to

\[
P_{SI}(k; y) := \sum_{M=1}^{\infty} P_{SI}(M) \frac{M!}{k!(M-k)!} \eta(y_i)^k (1 - \eta(y_i))^{M-k}
\]

\[
\rightarrow P_{SI}(k) \theta \left( \frac{L}{2} + y \right) \theta \left( \frac{L}{2} - y \right) + \delta_{k,0} \theta \left( -\frac{L}{2} + y \right) \theta \left( y - \frac{L}{2} \right)
\]

From this we have

\[
\bar{P}(k) := \int \frac{dy}{\Delta Y} \bar{P}_{SI}(k; y) \rightarrow \frac{L}{\Delta Y} \ P_{SI}(k) + \left( 1 - \frac{L}{\Delta Y} \right) \delta_{k,0}
\]

and

\[
\langle y^2 \rangle_k := \int \frac{dy}{\Delta Y} \bar{P}_{SI}(k; y) \ y^2
\]

\[
\rightarrow \left( \frac{L}{\Delta Y} \right) P_{SI}(k) \frac{L^2}{12} + \left( \frac{(\Delta Y)^2}{12} - \frac{L}{\Delta Y} \frac{L^2}{12} \right) \delta_{k,0}
\]

Notice that \( L/\Delta Y \) is the probability of a primary cluster being created within the row in question. We hence define

\[
\bar{\eta} := \frac{L}{\Delta Y}
\]

With this we have

\[
\left[ \tilde{N}_{\text{eff}} \right]^{-1} := \frac{12}{L^2} \sum_{N=1}^{\infty} P_{PT}(N) N \ \sum_{k_1, \ldots, k_N=2}^{N} \prod_{i=2}^{N} \left( \bar{P}(k_i) \right) \frac{k_i^2 \langle y^2 \rangle_{k_i}}{\left( \sum_{i=1}^{N} k_i \right)^2}
\]

\[
\rightarrow \sum_{N=1}^{\infty} P_{PT}(N) \ \sum_{k_1, \ldots, k_N=2}^{N} \prod_{i=2}^{N} \left( \bar{P}(k_i) \right) \frac{k_i^2 \langle y^2 \rangle_{k_i}}{\left( \sum_{i=1}^{N} k_i \right)^2}
\]

\[
= \sum_{N=1}^{\infty} P_{PT}(N) \ \sum_{N'=1}^{N} \bar{\eta}^{N'} (1 - \bar{\eta})^{N-N'} C_{N'} \ \sum_{k_1, \ldots, k_{N'} \geq N+1} \ \sum_{i=1}^{N'} \prod_{i=2}^{N'} \left( \bar{P}(k_i) \right) \frac{\sum_{i=1}^{N'} k_i^2 \langle y^2 \rangle_{k_i}}{\left( \sum_{i=1}^{N'} k_i \right)^2}
\]

where in the last line \( N' \) is the number of clusters created within the \( y \) range of the row in question. We can hence obtain

\[
\tilde{N}_{\text{eff}} \sim \left[ \frac{\left( \sum_{i=1}^{N'} \frac{M_i^2}{12} \right)}{\left( \sum_{i=1}^{N'} \frac{M_i}{12} \right)^2} \right]^{-1}
\]

in the short drift limit. Notice that the effective number of track electrons for the angle term is determined by primary ionization statistics for \( N' \) and secondary ionization statistics for \( M_i \). If \( L \) and hence \( <N'> \) is large enough for the approximation:

\[
\sum_{i=1}^{N'} M_i \approx N' \langle M \rangle =: N' \tilde{M}
\]

then we arrive at

\[
\lim_{z \to 0} N_{\text{eff}} = \left[ \frac{1}{N_{PT}} \right] \left[ \left( \frac{M}{\tilde{M}} \right)^2 \right]^{-1}
\]
This last formula has exactly the same form as the effect of gas gain fluctuation. Since the secondary ionization has a long tail and the number of primary electrons per row is less than 20 for L=6.3 mm, the approximation is expected to be bad. Nevertheless, the formula suggests that the effective number of electrons for the angle term is much smaller than that for the diffusion term as seen from the sample calculation.

[II] Long drift limit

In the long drift limit (\( \sigma_d/L \gg 1 \)), the diffusion dominates the row height and we have

\[
\eta(y) := \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dy'}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{(y'-y)^2}{2\sigma_d^2}\right)
\]
\[
\quad \rightarrow \frac{L}{\sqrt{2\pi}\sigma_d} \exp\left(-\frac{y^2}{2\sigma_d^2}\right) \quad \text{as} \quad \frac{\sigma_d}{L} \rightarrow \infty
\]

Notice that this probability is infinitesimal in the limit and hence we can safely ignore the probability of more than one electron from the same cluster reaching the row in question is negligible. We hence obtain

\[
\bar{P}_{SI}(k; y) := \sum_{M=1}^{\infty} P_{SI}(M) \frac{M!}{k!(M-k)!} \eta(y)^k (1-\eta(y))^{M-k} \\
\rightarrow \sum_{M=k}^{\infty} P_{SI}(M) [(1-\eta)^M \delta_{k,0} + M\eta (1-\eta)^{M-1}\delta_{k,1}] \\
\approx \sum_{M=k}^{\infty} P_{SI}(M) [(1-M\eta)\delta_{k,0} + M\eta\delta_{k,1}]
\]

leading us to

\[
\bar{P}_{SI}(k; y) \approx (1-M\eta(y))\delta_{k,0} + M\eta(y)\delta_{k,1}
\]

Using this we have

\[
\bar{P}(k) := \int \frac{dy}{\Delta Y} \bar{P}_{SI}(k; y) \rightarrow (1-M\bar{\eta})\delta_{k,0} + M\bar{\eta}\delta_{k,1}
\]

and

\[
\langle y^2 \rangle_k := \int \frac{dy}{\Delta Y} \bar{P}_{SI}(k; y) y^2 \\
\rightarrow \left(\frac{(\Delta Y)^2}{12} - M\bar{\eta}\sigma_d^2\right)\delta_{k,0} + M\bar{\eta}\sigma_d^2\delta_{k,1}
\]

with

\[
\bar{\eta} := \frac{L}{\Delta Y}
\]

where we have assumed that Delta Y is big enough compared to sigma_d. Using these formulae we can now calculate Neff for the angle term.
where \( K \) is the number of electrons arrived at the row in question. We can hence write

\[
[N_{\text{eff}}]^{-1} = \lim_{\sigma_d/L \to \infty} \frac{12 \sigma_d^2}{L^2} \left\langle \frac{1}{K} \right\rangle
\]

The effective number of track electrons for the angle term hence decreases with the drift length because of the increase of \( \sigma_d \) as seen in the sample calculation. As a matter of fact, we can rewrite the angle term in the large drift limit as

\[
\sigma_{\phi:\text{ang}}^2 \to \tan^2 \phi \left\langle \frac{1}{K} \right\rangle \sigma_d^2
\]

which indicates that diffusion dominates also in the y-direction. For convenience I show the same sample calculation here again. We see slight increase of the Neff at small drift distance due to de-clustering followed by monotonic decrease at longer drift distance. If you make the row too narrow, you cannot benefit because of the diffusion in the y direction. The figure suggests that we have to keep the diffusion for the maximum drift below 0.15. \( L=6.3 \text{mm} \) for the LC-TPC seems reasonable.
Angular Dependence of Neff:
Naively, Neff(\phi) is expected to be proportional to the track length and hence to scale as 1/\cos(\phi). In reality Neff increases more rapidly than 1/\cos(\phi) in particular at short drift distance where the edge effect for primary clusters near the pad row boundaries is more important and more rapidly diminishes with \phi.

Angular Dependence of Nhateff:
The situation is opposite for the effective number of primary clusters. Nhateff does not increase as rapidly as 1/\cos(\phi). This is because the chance for a primary cluster near pad row boundaries to have electrons migrating from one pad row to the other gets larger with \phi since the projected cluster-to-cluster becomes shorter.
**Gas Gain Fluctuation**

**Coordinate = Gain-Weighted Mean**

**PDF for gain-weighted mean**

We now switch on the gas gain fluctuation and assume that the coordinate measured by the readout plane is the gain-weighted mean of the $N$ ionization electrons.

**Primary ionization statistics**

$$G_{ij} = \frac{G_{ij}}{P_G}$$

where $Cd$ is the diffusion coefficient and $z$ is the drift length. $G_{ij}$ is the gas gain for $j$-the electron in $i$-th cluster whose P.D.F. is given by $P_G$ as before. Recall that the primary ionization statistics is governed by

$$P_{PI}(N) := \frac{\bar{N}^N}{N!} \exp(-\bar{N})$$

and effective cluster size distribution by

$$\bar{P}_{SI}(k; y_i) := \sum_{M=1}^{\infty} P_{SI}(M) \frac{M!}{k!(M-k)!} \eta(y_i)^k (1 - \eta(y_i))^{M-k}$$

where $k$ is the number of electrons that is accepted by the row in question starting from the $i$-th cluster created at $y=y_i$. Since $Y_i$ and $\Delta x_{ij}$ have no bias, we obviously have

$$\langle \bar{x} \rangle := \int d\bar{x} P(\bar{x}) \bar{x} = \bar{x}$$
Variance of the G.W.M.

By definition the variance of the gain-weighted mean is given by

$$\sigma^2_{\bar{x}} := \int d\bar{x} P(\bar{x}) (\bar{x} - \langle \bar{x} \rangle)^2$$

Substituting

$$\langle \bar{x} \rangle = \tilde{x}$$

we have

$$\sigma^2_{\bar{x}} = \left\langle \left( \frac{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij} (y_i \tan \phi + \Delta x_{ij})}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle$$

$$= \sigma_d^2 \left\langle \left( \frac{\sum_{i,j} G_{ij}^2}{\left( \sum_{i,j} G_{ij} \right)^2} \right)^2 \right\rangle + \tan^2 \phi \left\langle \left( \frac{\sum_{i=1}^{N} y_i \sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle$$

Making the same approximation

$$\sum_{i,j} G_{ij} \approx \bar{G} \sum_{i=1}^{N} k_i$$

as we did in the phi=0 case, we obtain

$$\sigma_{\bar{x},\text{diff}}^2 := \sigma_d^2 \left\langle \left( \frac{\sum_{i,j} G_{ij}^2}{\left( \sum_{i,j} G_{ij} \right)^2} \right)^2 \right\rangle \approx \sigma_d^2 \left\langle \frac{1}{\sum_{i} k_i} \right\rangle \left\langle \left( \frac{\bar{G}}{G} \right)^2 \right\rangle$$

for the angle-independent term which is none other than the formula we derived before for the phi=0 case.

For the angle-dependent term we have

$$\sigma_{\bar{x}:\text{ang}}^2 := \tan^2 \phi \left\langle \left( \frac{\sum_{i=1}^{N} y_i \sum_{j=1}^{k_i} G_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{ij}} \right)^2 \right\rangle$$

$$\approx \tan^2 \phi \left[ \left\langle \frac{\sum_i k_i^2 y_i^2}{(\sum_i k_i)^2} \right\rangle + \left\langle \frac{\sum_i y_i^2}{(\sum_i k_i)^2} \right\rangle \sigma^2_{\bar{G}} \right]$$

[I] Short drift limit

In the short drift limit, the angle term becomes

$$\sigma_{\bar{x}:\text{ang}}^2 \approx \tan^2 \phi \left[ \left\langle \frac{\sum_i M_i^2}{(\sum_i M_i)^2} \right\rangle + \left\langle \frac{1}{\sum_i M_i} \right\rangle \sigma^2_{\bar{G}} \right] \frac{L^2}{12}$$

since there is no diffusion in this limit and hence we can replace $k_i$ by $M_i$ and the average of $y_i^2$ becomes independent of $k_i$ and just gives $L^2/12$. The formula implies the effective number for the angle term being

$$\hat{N}_{\text{eff}} \approx \left[ \left\langle \frac{\sum_i M_i^2}{(\sum_i M_i)^2} \right\rangle + \left\langle \frac{1}{\sum_i M_i} \right\rangle \sigma^2_{\bar{G}} \right]^{-1}$$

where the 2nd term in the square bracket is from gain fluctuation.
We restart from

\[ \sigma_{x:ang}^2 := \tan^2 \phi \left\langle \left( \frac{\sum_{i=1}^{N} y_i \sum_{j=1}^{k_i} G_{i,j}}{\sum_{i=1}^{N} \sum_{j=1}^{k_i} G_{i,j}} \right)^2 \right\rangle \]

\[ \approx \tan^2 \phi \left[ \left\langle \frac{\sum_{i} k_i^2 y_i^2}{(\sum_{i} k_i)^2} \right\rangle + \left\langle \frac{\sum_{i} k_i y_i^2}{(\sum_{i} k_i)^2} \right\rangle \sigma^2 \left( \frac{G}{\bar{G}} \right) \right] \]

Recalling that in the long drift limit the row height becomes negligible and the effective cluster size \( k_i \) for the \( i \)-th cluster can be at most 1 and that each electron accepted by the row must have experienced average diffusion of \( \sigma_d^2 \), we have

\[ \sigma_{x:ang}^2 \approx \tan^2 \phi \left[ \left\langle \frac{1}{\sum_{i} k_i} \right\rangle + \left\langle \frac{1}{\sum_{i} k_i} \sigma^2 \left( \frac{G}{\bar{G}} \right) \right\rangle \right] \sigma_d^2 \]

\[ \approx \tan^2 \phi \left\langle \frac{1}{K} \right\rangle \left\langle \left( \frac{G}{\bar{G}} \right)^2 \right\rangle \sigma_d^2 \]

where \( K \) is the number of electrons arrived at the row in question. This matches our naive expectation from the \( \phi=0 \) formula for the \( x \)-resolution.

In the short drift region, the gas gain fluctuation only slightly reduces the Neff for the angle term (10% effect). Since the \( \sigma_d^2 \) will not exceed 0.2 for the LC-TPC, we can conclude that the angle term is rather insensitive to the gain fluctuation.
Finite Size Pads

Coordinate = Charge Centroid

PDF for charge centroid

We now replace the continuous readout plane with an array of finite size pads. The finite size pads break the translational symmetry. We hence need to specify the track position relative to the pad center. The arrival point of j-th ionization electron from i-th primary cluster is given by

\[ x_{ij} = \tilde{x} + y_i \tan \phi + \Delta x_{ij} \]

\[ y_{ij} = y_i + \Delta y_{ij} \]

The charge centroid is then given by

\[ \bar{x} = \frac{\sum_a Q_a (aw) / \sum_a Q_a} \]

with \( w \) being the pad pitch. The probability distribution function for charge centroid is

\[
P(\bar{\bm{x}}; \bar{x}) = \sum_{N=1}^{\infty} P_{P1}(N) \sum_{M_1, \ldots, M_N} \prod_{i=1}^{N} \left[ \int \frac{dy_i}{\Delta Y} P_{SI}(M_i) \int d\Delta x_{ij} P_D(\Delta x_{ij}; \sigma_d) \int d\Delta y_{ij} P_D(\Delta y_{ij}; \sigma_d) \int dG_{ij} P_G \left( \frac{G_{ij}}{G} \right) \prod_a \left[ \int d\Delta Q_a P_E(\Delta Q_a; \sigma_E) \right] \int dQ_a \delta \left( Q_a - \sum_{i=1}^{N} \sum_{j=1}^{M_i} G_{ij} F_a(x_{ij}, y_{ij}) - \Delta Q_a \right) \right] \times \delta \left( \bar{x} - \frac{\sum_a Q_a (aw)}{\sum_a Q_a} \right)
\]

The gas gain for the j-th ionization electron from the i-th primary cluster, and \( \Delta Q_a \) is the electronic noise:

\[ \langle (\Delta Q_a)^2 \rangle = \sigma_E^2 \]
Variance of charge centroid

In order to take into account the effect of finite size pads as known as the S-shape systematics, we define the variance by

\[
\sigma_x^2 := \int_{-\frac{1}{2}}^{\frac{1}{2}} d\left(\frac{\bar{x}}{w}\right) \int d\bar{x} P(\bar{x}; \bar{x}) (\bar{x} - \bar{x})^2
\]
as with the phi=0 case. Substituting the PDF given above in this and with some arithmetics, we obtain

\[
\sigma_x^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\left(\frac{\bar{x}}{w}\right) \left[ [A'] + [B'] \left\langle \left(\frac{G}{G}\right)^2 \right\rangle \right] + [C]
\]

with \([A'], [B'], [C]\) corresponding to \([A], [B], [C]\) for the phi=0 case. \([A']\) is independent of gas gain and given by

\[
[A'] := \sum_{a,b} (ab w^2) \left[ \left\langle \sum_i k_i^2 \left( \langle F_a \rangle \langle F_b \rangle \right)_{k_i} - \langle F_a \rangle_{k_i} \langle F_b \rangle_{k_i} \right\rangle \right] + \left\langle \left( \sum_i k_i \langle F_a \rangle_{k_i} \right) \left( \sum_i k_i \langle F_b \rangle_{k_i} \right) \right\rangle
\]

\[
+ \left\langle \left( \sum_i k_i \langle F_a \rangle_{k_i} \right) \left( \sum_i k_i \langle F_b \rangle_{k_i} \right) \right\rangle - \left\langle \sum_i k_i \langle F_a \rangle_{k_i} \right\rangle \left\langle \sum_i k_i \langle F_b \rangle_{k_i} \right\rangle
\]

\[
+ \sum_{a} (a w) \left\langle \sum_i k_i \langle F_a \rangle_{k_i} \right\rangle \left( \sum_i k_i \langle F_b \rangle_{k_i} \right) - \bar{x}
\]

and hence almost purely geometric when the 2nd term dominates in the R.H.S.

where we have defined

\[
\langle F_a \rangle := \int d\Delta x \, P_D(\Delta x; \sigma_d) \times F_a(\bar{x} + y \tan \phi + \Delta x)
\]

\[
\langle F_a \rangle_{k_i} := \frac{1}{P(k_i)} \int \frac{dy}{\Delta Y} P_{SI}(k; y) \langle F_a \rangle
\]

\[
\langle F_a \rangle \langle F_b \rangle_{k_i} := \frac{1}{P(k)} \int \frac{dy}{\Delta Y} P_{SI}(k; y) \langle F_a \rangle \langle F_b \rangle
\]

and the outermost average is taken as

\[
\langle \cdots \rangle := \sum_{N=1}^{\infty} P_{PI}(N) \sum_{k_1, \ldots, k_N} \prod_{i=1}^{N} \left( \bar{P}(k_i) \right) \langle \cdots \rangle
\]

In the de-clustering limit, only \(k_i=1\) counts and \([A']\) becomes

\[
\lim_{\sigma_d \to \infty} [A'] = \sum_{a,b} (ab w^2) \left( \langle F_a \rangle \langle F_b \rangle \right)_{k=1}
\]

\[
- \left( \sum_a (aw) \langle F_a \rangle_{k=1} \right)^2 \left( \sum_i k_i \right) - \left( \sum_{a} (a w) \langle F_a \rangle - \bar{x} \right)^2
\]

and hence almost purely geometric when the 2nd term dominates in the R.H.S.
[B'] is given by

$$[B'] := \sum_{a,b} (ab w^2) \left( \frac{\sum_i k_i (\langle \langle F_a F_b \rangle \rangle_{k_i} - \langle F_a \rangle \langle F_b \rangle_{k_i})}{(\sum_i k_i)^2} \right)$$

where

$$\langle \langle F_a F_b \rangle \rangle_{k} := \frac{1}{P(k)} \int \frac{dy}{\Delta Y} \tilde{P}_{SI}(k; y) \int d\Delta x P_D(\Delta x; \sigma_d) \times F_a(\bar{x} + y \tan \phi + \Delta x) F_b(\bar{x} + y \tan \phi + \Delta x)$$

This term represents the contributions from diffusion, gas gain fluctuation, and finite pad pitch. In the de-clustering limit, [B'] becomes

$$\lim_{\frac{\sigma_d}{L} \to \infty} [B'] := \sum_{a,b} (ab w^2) \left[ \langle \langle F_a F_b \rangle \rangle_{k=1} - \langle F_a \rangle \langle F_b \rangle_{k=1} \right] \times \left( \frac{1}{\sum_i k_i} \right)$$

The last term [C] is exactly as before and from the electronic noise contribution

$$[C] := \left( \frac{\sigma_E}{G} \right)^2 \left( \frac{1}{(\sum_i k_i)^2} \right) \sum_a (a w)^2$$

As far as the clustering effect is negligible, this term is independent of the drift length as before.

As noted before, the de-clustering limit is never reached in practice, since the pad row height is so chosen. It is hence useful to consider an asymptotic formula in such a case where the diffusion is large enough compared to the pad width while it is short enough compared to the pad height.

This part should be rewritten according to Ryo's thesis!
Comparison with Exp.

\[ \sigma^2 = \sigma_0^2 + \left( \frac{C_D}{\sqrt{N_{\text{eff}}}} \right)^2 z \]