An estimation of the influence of gain variation on the dE/dx resolution

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The influence of pad-row to pad-row gas gain variation on the dE/dx measurement was estimated with simple assumptions. The influence was found to be rather small and would be difficult to be detected.

I assumed the following.

1. The specific energy losses measured by pad rows are mutually independent.
2. The average gas gain is constant over a single pad row.
3. The relative variance of the gas gain is the same for all the pad rows.
4. The charge measurements of all the pad rows are used (100%-retained).
5. The contribution of electronic noise is negligible.

For a single pad row the measured charge (S_1) is given by

$$S_1 = \sum_{j=1}^{n} G_{1j} \tag{1}$$

$$\langle S_1 \rangle = \langle n \rangle \cdot \langle G_1 \rangle \tag{2}$$

where n is the number of drift electrons detected by the pad row, and G_{1j} is the gas gain for the j-th electron. The relative variance (R_1^2) is given by

$$
\begin{aligned}
R_1^2 &= \frac{\left\langle (S_1 - \langle S_1 \rangle)^2 \right\rangle}{\langle S_1 \rangle^2} && (3)\\
&= \frac{\left\langle \left(\sum_{j=1}^{n} G_{1j} - \langle n \rangle \cdot \langle G_1 \rangle \right)^2 \right\rangle}{\langle n \rangle^2 \cdot \langle G_1 \rangle^2} && (4)\\
&= \frac{\left\langle \left(\sum_{j=1}^{n} \left((G_{1j} - \langle G_1 \rangle) + \langle G_1 \rangle \cdot (n - \langle n \rangle) \right) \right)^2 \right\rangle}{\langle n \rangle^2 \cdot \langle G_1 \rangle^2} && (5)\\
&= \frac{\langle n \rangle \cdot \left\langle (G_1 - \langle G_1 \rangle)^2 \right\rangle + \langle G_1 \rangle^2 \cdot \left\langle (n - \langle n \rangle)^2 \right\rangle}{\langle n \rangle^2 \cdot \langle G_1 \rangle^2} && (6)\\
&= \frac{\langle n \rangle \cdot \sigma_{G_1}^2 + \langle G_1 \rangle^2 \cdot \sigma_n^2}{\langle n \rangle^2 \cdot \langle G_1 \rangle^2} && (7)\\
&= \frac{1}{\langle n \rangle} \cdot \left(\frac{\sigma_{G_1}^2}{\langle G_1 \rangle^2} + \frac{\sigma_n^2}{\langle n \rangle} \right) . && (8)
\end{aligned}
$$

For the measurement of monochromatic X-rays (^{55}Fe K$_\alpha$, for example) Eq. (8) gives

$$R_1^2 = \frac{f+F}{\langle n \rangle} \tag{9}$$

with f being the relative variance of gas gain and F, the Fano factor, if all the converted electrons are detected by the pad row.

For the average of N pad rows

$$\begin{aligned} S &= \frac{1}{N} \cdot \sum_{i=1}^{N} S_i &(10)\\ \langle S \rangle &= \frac{1}{N} \cdot \sum_{i=1}^{N} \langle S_i \rangle &(11)\\ &= \frac{\langle n \rangle}{N} \cdot \sum_{i=1}^{N} \langle G_i \rangle \ . &(12) \end{aligned}$$

Its variance is given by

$$\begin{aligned} \sigma_S^2 &= \left\langle (S - \langle S \rangle)^2 \right\rangle &(13)\\ &= \frac{1}{N^2} \cdot \left\langle \left(\sum_{i=1}^{N} (S_i - \langle S_i \rangle)^2 \right) \right\rangle &(14)\\ &= \frac{1}{N^2} \cdot \sum_{i=1}^{N} \sigma_{S_i}^2 &(15)\\ &= \frac{1}{N^2} \cdot \sum_{i=1}^{N} \left(\langle n \rangle \cdot \sigma_{G_i}^2 + \langle G_i \rangle^2 \cdot \sigma_n^2 \right) \ . &(16) \end{aligned}$$

Replacing G_i with $\langle G_i \rangle \cdot g_i$, with $\langle g_i \rangle = 1$,

$$\begin{aligned} \sigma_S^2 &= \frac{1}{N^2} \cdot \sum_{i=1}^{N} \left(\langle n \rangle \cdot \langle G_i \rangle^2 \cdot \sigma_g^2 + \langle G_i \rangle^2 \cdot \sigma_n^2 \right) &(17)\\ &= \frac{1}{N^2} \cdot \left(\langle n \rangle \cdot \sigma_g^2 + \sigma_n^2 \right) \cdot \sum_{i=1}^{N} \langle G_i \rangle^2 \ . &(18) \end{aligned}$$

The relative variance of S ($\equiv R^2$) is given by

$$\begin{aligned} R^2 &\equiv \frac{\sigma_S^2}{\langle S \rangle^2} &(19)\\ &= \frac{1}{\langle n \rangle} \cdot \left(\sigma_g^2 + \frac{\sigma_n^2}{\langle n \rangle} \right) \cdot \frac{\sum_{i=1}^{N} \langle G_i \rangle^2}{\left(\sum_{i=1}^{N} \langle G_i \rangle \right)^2} \ . &(20) \end{aligned}$$

Defining

$$\langle G_i \rangle = \langle G \rangle + \delta \langle G_i \rangle \ , \tag{21}$$

with

$$\langle G \rangle = \frac{1}{N} \cdot \sum_{i=1}^{N} \langle G_i \rangle \ , \tag{22}$$

The last factor in Eq. (20) is given by

$$\frac{\sum_{i=1}^{N} \langle G_i \rangle^2}{\left(\sum_{i=1}^{N} \langle G_i \rangle\right)^2} = \frac{\sum_{i=1}^{N} \left(\langle G \rangle + \delta \langle G_i \rangle\right)^2}{N^2 \cdot \langle G \rangle^2} \tag{23}$$

$$= \frac{1}{N^2} \cdot \frac{1}{\langle G \rangle^2} \cdot \sum_{i=1}^{N} \left(\langle G \rangle^2 + (\delta \langle G_i \rangle)^2\right) \tag{24}$$

$$= \frac{1}{N} \cdot \left(1 + \frac{1}{N} \cdot \frac{1}{\langle G \rangle^2} \cdot \sum_{i=1}^{N} (\delta \langle G_i \rangle)^2\right) \tag{25}$$

$$= \frac{1}{N} \cdot \left(1 + \frac{1}{N} \cdot \sum_{i=1}^{N} (\delta \Gamma_i)^2\right) , \tag{26}$$

with

$$\Gamma_i \equiv \frac{\langle G_i \rangle}{\langle G \rangle} . \tag{27}$$

Therefore

$$R^2 = \frac{1}{N} \cdot \frac{1}{\langle n \rangle} \cdot \left(\sigma_g^2 + \frac{\sigma_n^2}{\langle n \rangle}\right) \cdot (1 + \Delta)) , \tag{28}$$

with

$$\Delta \equiv \frac{1}{N} \cdot \sum_{i=1}^{N} (\delta \Gamma_i)^2 . \tag{29}$$

For example, if Γs are distributed evenly between $1 - d/2$ and $1 + d/2$, Γ_i is given by[1]

$$\Gamma_i = -\frac{d}{2} + \frac{d}{N} \cdot i \tag{30}$$

and

$$\Delta = \frac{1}{N} \cdot \sum_{i=1}^{N} \left(-\frac{d}{2} + \frac{d}{N} \cdot i\right)^2 \tag{31}$$

$$= \frac{d^2}{N} \cdot \sum_{i=1}^{N} \left(-\frac{1}{2} + \frac{i}{N}\right)^2 \tag{32}$$

$$= \frac{d^2}{N} \cdot \sum_{i=1}^{N} \left(\frac{1}{4} - \frac{i}{N} + \frac{i^2}{N^2}\right) \tag{33}$$

$$= \frac{d^2}{N} \cdot \left(\frac{N}{4} - \frac{1}{N} \cdot \sum_{i=1}^{N} i + \frac{1}{N^2} \cdot \sum_{i=1}^{N} i^2\right) \tag{34}$$

$$= d^2 \cdot \left(\frac{1}{4} - \frac{1}{2} \cdot \frac{N+1}{N} + \frac{1}{6} \cdot \frac{(N+1)(2N+1)}{N^2}\right) . \tag{35}$$

The last equation gives 0.0133 (0.0134) for the value of Δ for $N = 220$ (26) with $d = 0.2$.

[1]The smallest value of Γ_i is set to $1 - d + d/N$ instead of $1 - d$ just to simplify the subsequent equations. The result is scarcely affected for $N \gg 1$.

If Γs are randomly selected from a uniform population between $1 - d/2$ and $1 + d/2$

$$R^2 = \frac{1}{\langle n \rangle} \cdot \left(\sigma_g^2 + \frac{\sigma_n^2}{\langle n \rangle} \right) \cdot \frac{1}{N} \cdot \left(1 + \frac{N-1}{N} \cdot \sigma_\Gamma^2 \right) \tag{36}$$

$$= \frac{1}{N} \cdot \frac{1}{\langle n \rangle} \cdot \left(\sigma_g^2 + \frac{\sigma_n^2}{\langle n \rangle} \right) \cdot (1 + \Delta')) \ , \tag{37}$$

with

$$\Delta' \equiv \frac{N-1}{N} \cdot \sigma_\Gamma^2 \ . \tag{38}$$

The last expression gives 0.0133 (0.0128) for the value of Δ' for $N = 200$ (28) with $d = 0.2$ ($\sigma_\Gamma^2 = (0.4)^2/12$).

Consequently, the degradation of the $\mathrm{d}E/\mathrm{d}x$ resolution due to the pad-row to pad-row gas-gain variation by a factor of $\sqrt{1 + \Delta}$ or $\sqrt{1 + \Delta'}$ is expected to be too small to be detected under the conditions mentioned at the beginning.